For semilinear PDEs in *two* variables *x, y*, there exists a change of coordinate from *(x, y)* to *(μ, η)* such that:

(i) a hyperbolic PDE becomes \( u_{\mu\eta} = \tilde{d}(\mu, \eta, u, u_{\mu}, u_{\eta}) \); accomplish this by finding the two families of characteristics as \( \mu(x, y) = \text{const} \) and \( \eta(x, y) = \text{const} \), then take the new coordinates to be \( \mu = \mu(x, y) \) and \( \eta = \eta(x, y) \).

(ii) a parabolic PDE becomes \( u_{\mu\mu} = \tilde{d}(\mu, \eta, u, u_{\mu}, u_{\eta}) \); accomplish this by finding the family of characteristics as \( \mu(x, y) = \text{const} \), then take the first new coordinate to be \( \mu = \mu(x, y) \) and choose \( \eta \) to be anything at all that is transverse to \( \mu \).

(iii) an elliptic PDE becomes \( u_{\mu\mu} + u_{\eta\eta} = \tilde{d}(\mu, \eta, u, u_{\mu}, u_{\eta}) \); accomplish this as in Exercise 2.2.10.


This reduction to the (much simpler) canonical form of the PDE generally does not work when the PDE has more than 2 variables. Reason: a PDE with *k* variables has \((k^2 - k)/2\) mixed second derivative terms whose coefficients we want to transform to 0, and has \(k - 1\) pure second partial derivatives whose coefficients we want to specify; but we have only \(k\) “change of coordinate” functions to play with, and \((k^2 - k)/2 + k - 1 > k\) when \(k > 2\).

All the same, the canonical form does sometimes help solve the equation, when there are only two variables. MCOWEN has several examples.

**IMPORTANT EXAMPLE**

For the wave equation \( u_{xx} - u_{yy} = 0 \), the characteristics are \( y = x + \text{const} \) and \( y = -x + \text{const} \) (check this; it’s easy). Following (i) above, we let \( \mu = x + y \) and \( \eta = x - y \). Then

\[
\begin{align*}
u_x &= u_{\mu} + u_\eta, \\
u_y &= u_{\mu} - u_\eta,
\end{align*}
\]

and

\[
\begin{align*}
u_{xx} &= u_{\mu\mu} + u_{\mu\eta} + u_{\eta\mu} + u_{\eta\eta}, \\
u_{yy} &= u_{\mu\mu} - u_{\mu\eta} - u_{\eta\mu} + u_{\eta\eta},
\end{align*}
\]

so that the wave equation transforms to \( 0 = u_{xx} - u_{yy} = 4u_{\mu\eta} \). Because \( u_{\mu\eta} = 0 \), we deduce by integrating twice that

\[
u = F(\mu) + G(\eta)
\]

for arbitrary functions \( F \) and \( G \). Hence the wave equation has solution

\[
u(x, y) = F(x + y) + G(x - y).
\]

Check directly that this solves the wave equation, when \( F \) and \( G \) are twice differentiable!