Material: Poisson’s equation in 3-dimensions.

- Suppose we want to solve the Poisson equation
  \[ \Delta u(\vec{x}) = -4\pi f(\vec{x}), \quad \vec{x} \in \mathbb{R}^3, \]
  with \( u(\vec{x}) \to 0 \) as \( |\vec{x}| \to \infty \). Assume \( f(\vec{x}) = 0 \) for all large \( |\vec{x}| \). (Think of \( f \) as describing the mass density of a planet — we’ll justify that interpretation later.)
- First, we take \( k = 1 \) and solve the 3-dimensional diffusion equation
  \[ v_t = \Delta v, \quad \vec{x} \in \mathbb{R}^3, \quad \text{with initial value} \quad v(\vec{x}, 0) = 4\pi f(\vec{x}), \]
  by writing
  \[
v(\vec{x}, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(x - x^*, t)S(y - y^*, t)S(z - z^*, t)4\pi f(x^*, y^*, z^*) \, dx^* dy^* dz^*
  \]
  where \( S \) is the source function for the diffusion equation in one dimension. (See Exercise 2.4.19 for the justification.)
- We claim the desired solution of Poisson’s equation is
  \[ u(\vec{x}) = \int_0^\infty v(\vec{x}, t) \, dt. \] (1)

**Proof.** The formula for \( u \) gives
  \[
  \Delta u(\vec{x}) = \int_0^\infty \Delta v(\vec{x}, t) \, dt \\
  = \int_0^\infty v_t(\vec{x}, t) \, dt \\
  = v(\vec{x}, \infty) - v(\vec{x}, 0) \\
  = 0 - 4\pi f(\vec{x}) \quad \text{since} \quad v(\vec{x}, t) \to 0 \quad \text{as} \quad t \to \infty.
  \]
  And \( u(\vec{x}) \to 0 \) as \( |\vec{x}| \to \infty \) because \( S(x - x^*, t) \to 0 \) as \( x \to \pm \infty \), and so on. \( \Box \)

- Next we evaluate our formula for \( u \): by substituting the definition of \( v \) into the formula for \( u \), we find
  \[
u(\vec{x}, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(x - x^*, t)S(y - y^*, t)S(z - z^*, t) \, dt \, 4\pi f(x^*, y^*, z^*) \, dx^* dy^* dz^*.
  \]
We can evaluate the $t$-integral:
\[
\int_0^\infty S(x - x^*, t)S(y - y^*, t)S(z - z^*, t) \, dt
\]
\[
= \int_0^\infty \frac{1}{(4\pi t)^{3/2}} e^{-|\vec{x} - \vec{x}^*|^2/4t} \, dt \quad \text{using the definition of } S
\]
\[
= \int_0^\infty \frac{1}{(4\pi s)^{3/2}} e^{-1/4s} ds \cdot \frac{1}{|\vec{x} - \vec{x}^*|} \quad \text{where } t = s|\vec{x} - \vec{x}^*|^2
\]
\[
= \frac{1}{2\pi^{3/2}} \int_0^\infty e^{-r^2} dr \cdot \frac{1}{|\vec{x} - \vec{x}^*|} \quad \text{where } s = 1/4r^2
\]
\[
= \frac{1}{4\pi |\vec{x} - \vec{x}^*|}.
\]

Hence we conclude
\[
u(\vec{x}) = \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{1}{|\vec{x} - \vec{x}^*|} f(x^*, y^*, z^*) \, dx^* \, dy^* \, dz^*
\]
is the solution of Poisson’s equation
\[
\Delta u(\vec{x}) = -4\pi f(\vec{x}), \quad \vec{x} \in \mathbb{R}^3,
\]
in 3-dimensions, with $u(\vec{x}) \to 0$ as $|\vec{x}| \to \infty$.

Aside. In 1 or 2 dimensions, the analogous derivation does not work, because the integral in (1) is typically infinite in value. But the end result for $u$ still holds, except with $1/|\vec{x} - \vec{x}^*|$ replaced by $-|x - x^*|$ (in dimension 1) or $\log 1/|\vec{x} - \vec{x}^*|$ (in dimension 2). The constant $4\pi$ in Poisson’s equation also must change, to 2 (in dimension 1) or $2\pi$ (in dimension 2).