Total: 100 points. Do 4 out of 5 questions. You MUST do #5. EXPLAIN every answer. No books, notes, calculators or computers allowed on this test.

1 (25 points). Deduce all solutions $x$ of:

$$x \equiv 1 \mod 3$$

and

$$x \equiv 2 \mod 4.$$  

Guessing is not acceptable: you should use the method of the Chinese Remainder Theorem, or some similar technique.

**Solution 1:**

Since $x \equiv 1 \mod 3$, we have $x = 1 + 3k$ for some $k \in \mathbb{Z}$. But $x \equiv 2 \mod 4$ and so $1 + 3k \equiv 2 \mod 4$, or $3k \equiv 1 \mod 4$. Multiplying by 3 gives $9k \equiv 3 \mod 4$, which reduces to $k \equiv 3 \mod 4$ (since $9k \equiv k \mod 4$). That is, $k = 3 + 4\ell$ for some $\ell \in \mathbb{Z}$.

Thus $x = 1 + 3k = 10 + 12\ell$. So $x$ equals 10 plus a multiple of 12. The set of all solutions is \( \{ x = 10 + 12m : m \in \mathbb{Z} \} \).

**Solution 2:**

Following the proof of the Chinese Remainder Theorem, we set

$$N = 3 \cdot 4 = 12,$$

$$N_1 = 4,$$

$$N_2 = 3.$$

We want to find solutions of

$$N_1y_1 \equiv 1 \mod 3,$$

that is, $4y_1 \equiv 1 \mod 3$,

$$N_2y_2 \equiv 1 \mod 4,$$

that is, $3y_2 \equiv 1 \mod 4$.

We see that $y_1 = 1$ and $y_2 = 3$ will do. So then one solution is

$$x = a_1N_1y_1 + a_2N_2y_2 = 1 \cdot 4 \cdot 1 + 2 \cdot 3 \cdot 3 = 22.$$

The Chinese Remainder Theorem tells us the solutions are hence the numbers of the form $x = 22 + 12m$ for $m \in \mathbb{Z}$ (using here that $N = 12$).
There are two boxes, each containing a huge number of colored balls:

- in Box 1, 70% of the balls are orange and 30% are blue;
- in Box 2, 30% of the balls are orange and 70% are blue.

(a) You reach into Box 1 and randomly take out 12 balls. Explain why the probability of getting 8 orange balls and 4 blue balls is \( p_1 = \binom{12}{8} (0.7)^8 (0.3)^4 \).

**Solution:** Each possible ordering of choosing 8 orange balls out of 12 will occur with probability \((0.7)^8 (0.3)^4\), because on each choice, you get an orange ball with probability 0.7 and a blue ball with probability 0.3. Further, there are \( \binom{12}{8} \) different ways to choose the 8 orange balls out of 12.

(b) You reach into Box 2 and randomly take out 12 balls. Explain why the probability of getting 8 orange balls and 4 blue balls is \( p_2 = \binom{12}{8} (0.3)^8 (0.7)^4 \).

**Solution:** Each possible ordering of choosing 8 orange balls out of 12 will occur with probability \((0.3)^8 (0.7)^4\), because on each choice, you get an orange ball with probability 0.3 and a blue ball with probability 0.7. Further, there are \( \binom{12}{8} \) different ways to choose the 8 orange balls out of 12.

(c) Finally, you reach into a box (not knowing which box it is) and randomly take out 12 balls. You get 8 orange balls and 4 blue balls. Find the probability that you reached into Box 1.

**Solution:**

The conditional probability of having reached into Box 1 given that you got 8 orange balls and 4 blue balls is:

\[
P(\text{you reached into Box 1} | \text{you got 8 orange balls and 4 blue balls}) = \frac{P(\text{you reached into Box 1 and you got 8 orange balls and 4 blue balls})}{P(\text{you got 8 orange balls and 4 blue balls})}
\]

\[
= \frac{\frac{1}{2} p_1}{\frac{1}{2} p_1 + \frac{1}{2} p_2}
\]

since you have probability \( \frac{1}{2} \) of reaching into each of Box 1 and Box 2

\[
= \frac{\frac{1}{2} \binom{12}{8} (0.7)^8 (0.3)^4}{\frac{1}{2} \binom{12}{8} (0.7)^8 (0.3)^4 + \frac{1}{2} \binom{12}{8} (0.3)^8 (0.7)^4} = \frac{(0.7)^4}{(0.7)^4 + (0.3)^4} \approx 96.7\%.
\]
3 (25 points). Fix \( n, k \in \mathbb{N} \). Suppose that \( n \) pairs of socks are put into the laundry, with each sock having one mate. The laundry machine randomly eats socks; a random set of \( k \) socks returns. Determine the expected number of complete pairs of returned socks.

Hints:
1. Here \( S \) is the set of all \( k \)-element subsets of the total set of \( 2n \) socks. It is assumed that each outcome is equally likely.
2. For each \( i = 1, \ldots, n \), let \( X_i \) be a random variable on \( S \) that equals 1 if both socks in the \( i \)-th pair are returned, and 0 otherwise. The question is asking you to find \( E(X_1 + X_n) \).
3. Use the linearity of expectation.
4. Show \( E(X_i) = P(X_i = 1) \) the probability that the \( i \)-th pair is returned.
5. Evaluate \( P(X_i = 1) \).

Solution:
Let \( S \) be the possibility space, consisting of \( k \)-sock subsets of the total \( 2n \) socks. This space has \( \binom{2n}{k} \) elements, each occurring with probability \( 1/(\binom{2n}{k}) \).

Let \( X \) be the number of pairs returned. We wish to calculate \( E(X) \).

Number the pairs 1 through \( n \), and let \( X_i \) be the random variable on \( S \) which is 1 if the \( i \)-th pair is returned, and 0 otherwise. Then the number of pairs returned is
\[
X = X_1 + \cdots + X_n,
\]
and so
\[
E(X) = E(X_1 + \cdots + X_n) = E(X_1) + \cdots + E(X_n)
\]
by the linearity of expectation.

The number of \( k \)-sock sets that contain the \( i \)-th pair is
\[
\binom{2n-2}{k-2},
\]
because after choosing the 2 socks from the \( i \)-th pair, we still have to choose another \( k - 2 \) socks from the remaining \( 2n - 2 \) socks. Thus the probability of getting the \( i \)-th pair returned is
\[
P(X_i = 1) = \frac{\binom{2n-2}{k-2}}{\binom{2n}{k}}.
\]

Now, the expected value of \( X_i \) is
\[
E(X_i) = 0 \cdot P(X_i = 0) + 1 \cdot P(X_i = 1)
= P(X_i = 1)
= \frac{\binom{2n-2}{k-2}}{\binom{2n}{k}},
\]
and so the expected value of \( X \) is
\[
E(X) = n \frac{\binom{2n-2}{k-2}}{\binom{2n}{k}}.
\]
4 (25 points).
(a) Prove: “Suppose $a_n \to L$ and $b_n \to M$, and that $a_n \leq b_n$ for all $n$. Then $L \leq M$.”
(b) Disprove, by finding a counterexample: “Suppose $a_n \to L$ and $b_n \to M$, and that $a_n < b_n$ for all $n$. Then $L < M$.”

Solution:

(a) Let $c_n = a_n - b_n$, so that $c_n \leq 0$ for all $n$. Then $\lim c_n \leq 0$, by Lemma 13.17. But $\lim c_n = \lim a_n - \lim b_n = L - M$, so $L - M \leq 0$. That is, $L \leq M$.

Or: Let $\epsilon > 0$. Let $N \in \mathbb{N}$ be such that $|a_n - L| < \epsilon$ and $|b_n - M| < \epsilon$ for all $n \geq N$. Then
\[
L - \epsilon < a_n \leq b_n < M + \epsilon \quad \text{for all } n \geq N.
\]
This implies $L - \epsilon < M + \epsilon$, so that $L - M < 2\epsilon$. This holds for all $\epsilon > 0$, and so necessarily $L - M \leq 0$. That is, $L \leq M$.

Or: [Contradiction] Suppose instead that $L > M$. Let $\epsilon = \frac{L - M}{2}$, so that $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ and $|b_n - M| < \epsilon$ for all $n \geq N$. Then for all $n \geq N$,
\[
a_n > L - \epsilon = \frac{L + M}{2} = M + \epsilon > b_n.
\]
But this contradicts the assumption that $a_n \leq b_n$ for all $n$. Hence actually $L \leq M$.

(b) Let $a_n = \frac{1}{n}$ and $b_n = \frac{2}{n}$. Then $a_n < b_n$ for all $n$ but $L = M$ since: $L = \lim a_n = 0$ and $M = \lim b_n = 0$. 


5 (25 points). Prove the following statement (which is part of the Monotone Convergence Theorem):

“Every bounded nondecreasing sequence converges to its supremum.”

Solution:

Let \( \langle a \rangle \) be a bounded nondecreasing sequence. Since \( \langle a \rangle \) is bounded, it has an upper bound. Hence it has a supremum, by Completeness of \( \mathbb{R} \). Write \( \alpha = \sup \langle a \rangle \) for this supremum.

We wish to show \( a_n \to \alpha \). Let \( \varepsilon > 0 \). Notice \( \alpha - \varepsilon \) is smaller than the least upper bound \( \alpha \), and so it is not an upper bound for the sequence. Thus there exists some term of the sequence that is bigger than \( \alpha - \varepsilon \), say \( a_N > \alpha - \varepsilon \).

Since the sequence \( \langle a \rangle \) is nondecreasing, we know \( a_n \geq a_N \) for all \( n \geq N \). Hence \( a_n > \alpha - \varepsilon \) for all \( n \geq N \). But also \( a_n \leq \alpha \) for all \( n \), because \( \alpha \) is the supremum. Hence

\[-\varepsilon < a_n - \alpha \leq 0 \quad \text{for all } n \geq N.\]

This implies \( |a_n - \alpha| < \varepsilon \) for all \( n \geq N \), and so \( a_n \to \alpha \) as desired.