Solutions

MATH 247 — FALL 2000 — TEST 1

NAME:

Total: 100 points. Do 5 out of 6 questions. You **MUST** do #6. EXPLAIN every answer.
No books, notes, calculators or computers allowed on this test.

1 (20 points). Let $S = \{x \in \mathbb{R} : x^2 > 2x + 8\}$ and $T = \{x \in \mathbb{R} : x > 4\}$. Are the following statements true or false?

(a) $T \subseteq S$

(b) $S \subseteq T$

**Solution:** The key to the problem is to factor the quadratic:

$$x^2 - 2x - 8 = (x - 4)(x + 2).$$

Using this idea, we see that

$$S = \{x \in \mathbb{R} : x^2 > 2x + 8\}$$

$$= \{x \in \mathbb{R} : x^2 - 2x - 8 > 0\}$$

$$= \{x \in \mathbb{R} : (x - 4)(x + 2) > 0\}$$

$$= \{x \in \mathbb{R} : x > 4 \text{ or } x < -2\}.$$

So $T \subseteq S$ is true, because every number $x > 4$ does belong to $S$.

But $S \subseteq T$ is false, since the numbers $x < -2$ belong to $S$ but not to $T$. [For example, $-3 \in S$ because $(-3)^2 > 2(-3) + 8$. But $-3 \notin T$.]
2 (20 points). Without using words of negation (such as “no”, “not”, . . . ), write the negations of the following statements.

a) For all real numbers A there is an x < A such that f(x) > B.

b) There exists c ∈ R such that for all real numbers x, y ≥ c, if x > y then f(x) > f(y).

Solution:

a) In symbols, part a) says

\((\forall A \in \mathbb{R})(\exists x < A)f(x) > B.\)

To negate this, we change \(\forall\) to \(\exists\), change \(\exists\) to \(\forall\), and negate the statement at the end, getting:

\((\exists A \in \mathbb{R})(\forall x < A)\neg(f(x) > B).\)

But \(\neg(f(x) > B)\) means \(f(x) \leq B\), and so the desired negation is

\((\exists A \in \mathbb{R})(\forall x < A)f(x) \leq B.\)

In words, one could write:

For some real number A, for all x < A we have f(x) ≤ B.

b) In symbols, part b) says

\((\exists c \in \mathbb{R})(\forall x, y \geq c)(x > y \implies f(x) > f(y)).\)

To negate this, we change \(\forall\) to \(\exists\), change \(\exists\) to \(\forall\), and negate the statement at the end, getting:

\((\forall c \in \mathbb{R})(\exists x, y \geq c)\neg(x > y \implies f(x) > f(y)).\)

But if it is not true that \((x > y \implies f(x) > f(y))\), then we must have \(x > y\) and \(f(x) \leq f(y)\).

Hence the desired negation is

\((\forall c \in \mathbb{R})(\exists x, y \geq c)(x > y \text{ and } f(x) \leq f(y)).\)

In words, one could write:

For each c ∈ R there exists x, y ≥ c such that x > y and f(x) ≤ f(y).
3 (20 points). Let
\[ f(x) = \frac{x^2 - 1}{x^2 + 4}, \quad x \in \mathbb{R}. \]
Show that the image of \( f \) is \([-\frac{1}{4}, 1)\).

Solution: Write \( f(\mathbb{R}) \) for the image of \( f \).

i) \( f(\mathbb{R}) \subseteq [-\frac{1}{4}, 1) \).

Proof: For every \( x \in \mathbb{R} \), we have
\[ f(x) = \frac{x^2 - 1}{x^2 + 4} < 1 \quad \text{because} \quad x^2 - 1 < x^2 + 4 \]
(this last inequality simplifies to the true inequality \(-1 < 4\)). And also
\[ f(x) = \frac{x^2 - 1}{x^2 + 4} \geq -\frac{1}{4} \quad \text{because} \quad x^2 - 1 \geq -\frac{1}{4}(x^2 + 4) \]
(this last inequality simplifies to the true inequality \(\frac{3}{4}x^2 \geq 0\)).

This proves that \(-\frac{1}{4} \leq f(x) < 1 \) for all \( x \), and so \( f(\mathbb{R}) \subseteq [-\frac{1}{4}, 1) \).

ii) \( f(\mathbb{R}) \supseteq [-\frac{1}{4}, 1) \).

Proof: Let \( y \in [-\frac{1}{4}, 1) \). We want to show there exists an \( x \in \mathbb{R} \) with \( f(x) = y \), because then we will know \( y \in f(\mathbb{R}) \). In fact we can find two such \( x \)-values:
\[ f(x) = y \iff \frac{x^2 - 1}{x^2 + 4} = y \]
\[ \iff x^2 - 1 = y(x^2 + 4) \]
\[ \iff (1 - y)x^2 = 1 + 4y \]
\[ \iff x^2 = \frac{1 + 4y}{1 - y} \]
\[ \iff x = \pm \sqrt{\frac{1 + 4y}{1 - y}} \]
It is OK to take the square root here because the number inside is nonnegative: \( 1 + 4y \geq 0 \) because \( y \geq -\frac{1}{4} \), and \( 1 - y > 0 \) because \( y < 1 \).

This completes the proof that \( f(\mathbb{R}) = [-\frac{1}{4}, 1) \).
Let $n \geq 3$. Prove by induction that every set of $n$ elements has $\frac{1}{2}n(n - 1)$ subsets of size 2.

[For example, the set $A = \{x_1, x_2, x_3\}$ has the following subsets of size two: \{x_1, x_2\}, \{x_1, x_3\} and \{x_2, x_3\}. Here $n = 3$, and notice $\frac{1}{2}n(n - 1) = 3$, which correctly gives the number of subsets of size two. This proves your induction basis.]

**Solution:**

Write $P(n)$ for the statement that “every set of $n$ elements has $\frac{1}{2}n(n - 1)$ subsets of size 2”.

(a) [Basis step] Let $n = 3$. Then $P(3)$ is true, as shown in the statement of the problem.

(b) [Induction step] Assume $P(n)$ is true for $n = k$, so that every set of $k$ elements has $\frac{1}{2}k(k - 1)$ subsets of size 2. Let $A$ be a set of $k + 1$ elements. To complete the induction step, we need to show that $A$ has

$$\frac{1}{2}(k + 1)k$$

subsets of size 2, since that is the statement $P(n)$ with $n = k + 1$.

Write $A = \{x_1, x_2, \ldots, x_k, x_{k+1}\}$ for our set of $k + 1$ elements. Obviously there are $k$ subsets of size 2 that contain $x_{k+1}$, namely the subsets \{x_1, x_{k+1}\}, \{x_2, x_{k+1}\}, \ldots, \{x_k, x_{k+1}\}. And the subsets of size 2 that do not contain $x_{k+1}$ are precisely the subsets of \{x_1, x_2, \ldots, x_k\} of size 2. There are $\frac{1}{2}k(k - 1)$ such subsets by the induction hypothesis. Adding up, we find the number of subsets of $A$ of size 2 is:

$$\frac{1}{2}k(k - 1) + k = \frac{1}{2}k(k - 1 + 2) = \frac{1}{2}(k + 1)k,$$

exactly as we needed to show.
5 (20 points). For \( n \geq 2 \), find and prove a formula for \( \prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) \).

**Solution:** First we investigate numerically:

\[
\begin{align*}
n = 2 & \implies \prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = (1 - \frac{1}{2^2}) = \frac{3}{4} \\
n = 3 & \implies \prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = (1 - \frac{1}{2^2})(1 - \frac{1}{3^2}) = \frac{2}{3} \\
n = 4 & \implies \prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = (1 - \frac{1}{2^2})(1 - \frac{1}{3^2})(1 - \frac{1}{4^2}) = \frac{5}{8} \\
n = 5 & \implies \prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = (1 - \frac{1}{2^2})(1 - \frac{1}{3^2})(1 - \frac{1}{4^2})(1 - \frac{1}{5^2}) = \frac{3}{5}
\end{align*}
\]

So the results for \( n = 2, 3, 4, 5 \) are \( \frac{3}{4}, \frac{2}{3}, \frac{5}{8}, \frac{3}{5} \). There is no really obvious pattern here. But wait! \( \frac{2}{3} = \frac{4}{6} \) and \( \frac{3}{5} = \frac{6}{10} \). So the list can be rewritten as \( \frac{3}{4}, \frac{4}{6}, \frac{5}{8}, \frac{6}{10} \). Now we see that the \( n \)th term matches up with \( \frac{n+1}{2n} \), at least for the cases \( n = 2, 3, 4, 5 \).

Now we prove our conjecture that

\[
\prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}
\]

for all \( n \geq 2 \).

**Proof.**

(a) [Basis step] The first case to consider here is \( n = 2 \) (notice we don’t consider \( n = 1 \) in this problem). We already did the basis step above: when \( n = 2 \) we have \( \prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = (1 - \frac{1}{2^2}) = \frac{3}{4} = \frac{n+1}{2n} \).

(b) [Induction step] Let \( k \geq 2 \). Assume the formula holds for \( n = k \), so that

\[
\prod_{i=2}^{k} \left(1 - \frac{1}{i^2}\right) = \frac{k+1}{2k}.
\]

Then

\[
\prod_{i=2}^{k+1} \left(1 - \frac{1}{i^2}\right) = \left[1 - \frac{1}{(k+1)^2}\right] \prod_{i=2}^{k} \left(1 - \frac{1}{i^2}\right)
\]

by splitting of the last term of the product

\[
= \left[1 - \frac{1}{(k+1)^2}\right] \frac{k+1}{2k}
\]

by the induction hypothesis

\[
= \frac{(k+1)^2 - 1}{(k+1)^2} \frac{k+1}{2k}
\]

\[
= \frac{k^2 + 2k}{2k(k+1)}
\]

\[
= \frac{k + 2}{2(k+1)},
\]

which is the desired formula with \( n = k + 1 \). This proves the induction step. \( \square \)
6 (20 points). [You MUST do this problem.] Let \( f : \mathbb{R} \to \mathbb{R} \) be a function such that
\[
f(x + y) = f(x) + f(y)
\]
for all \( x, y \in \mathbb{R} \).

(i) Prove that \( f(0) = 0 \).
(ii) Let \( s \in \mathbb{R} \). Prove by induction that \( f(ns) = nf(s) \) for all \( n \in \mathbb{N} \).
(iii) Let \( t \in \mathbb{R} \). Deduce using part (ii) that \( f \left( \frac{m}{n} t \right) = \frac{m}{n} f(t) \) for all \( m, n \in \mathbb{N} \).

Solution:

(i) Choosing \( x = 0, y = 0 \), the given formula becomes
\[
f(0 + 0) = f(0) + f(0).
\]
That is, \( f(0) = f(0) + f(0) \). Subtracting \( f(0) \) from both sides of this equation gives \( 0 = f(0) \), as needed.

(ii) Write \( P(n) \) for the statement that \( f(ns) = nf(s) \).
[Basis step] With \( n = 1 \) the statement \( P(1) \) says \( f(1s) = 1f(s) \), or \( f(s) = f(s) \), which is true.
[Induction step] Assume \( P(n) \) is true for \( n = k \), so that \( f(ks) = kf(s) \). Then
\[
f((k + 1)s) = f(k + 1s)
\]
\[
= f(ks + s)
\]
\[
= f(ks) + f(s) \quad \text{by the given equation applied with } x = ks, y = s
\]
\[
= kf(s) + f(s) \quad \text{by our induction hypothesis}
\]
\[
= (k + 1)f(s).
\]
That is, we have proved \( P(n) \) is true with \( n = k + 1 \), completing the induction proof.

(iii) We have
\[
nf(\frac{m}{n} t) = f(\frac{m}{n} nt) \quad \text{by applying part (ii) with } s = \frac{m}{n} t
\]
\[
= f(nt)
\]
\[
= mf(t) \quad \text{by applying part (ii) with } s = t.
\]
Dividing through by \( n \) gives \( f(\frac{m}{n} t) = \frac{m}{n} f(t) \), as we wanted.