1 (20 points). Let $S = \{ x \in \mathbb{R} : x^2 > 2x + 8 \}$ and $T = \{ x \in \mathbb{R} : x > 4 \}$. Are the following statements true or false?
(a) $T \subseteq S$
(b) $S \subseteq T$
2 (20 points). Without using words of negation (such as “no”, “not”,…), write the negations of the following statements.

a) For all real numbers $A$ there is an $x < A$ such that $f(x) > B$.

b) There exists $c \in \mathbb{R}$ such that for all real numbers $x, y \geq c$, if $x > y$ then $f(x) > f(y)$. 
3 (20 points). Let
\[ f(x) = \frac{x^2 - 1}{x^2 + 4}, \quad x \in \mathbb{R}. \]
Show that the image of \( f \) is \([\frac{1}{4}, 1)\).
4 (20 points). Let $n \geq 3$. Prove by induction that every set of $n$ elements has $\frac{1}{2}n(n - 1)$ subsets of size 2.

[For example, the set $A = \{x_1, x_2, x_3\}$ has the following subsets of size two: $\{x_1, x_2\}, \{x_1, x_3\}$ and $\{x_2, x_3\}$. Here $n = 3$, and notice $\frac{1}{2}n(n - 1) = 3$, which correctly gives the number of subsets of size two. This proves your induction basis.]
5 (20 points). For $n \geq 2$, find and prove a formula for $\prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right)$. 
6 (20 points). [You MUST do this problem.] Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

\[ f(x + y) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R}. \]

(i) Prove that $f(0) = 0$.

(ii) Let $s \in \mathbb{R}$. Prove by induction that $f(ns) = nf(s)$ for all $n \in \mathbb{N}$.

(iii) Let $t \in \mathbb{R}$. Deduce using part (ii) that $f(\frac{m}{n} t) = \frac{m}{n} f(t)$ for all $m, n \in \mathbb{N}$. 