1. (25 points)
   (a) [8 points] A function \( f(x) \) on \([a, b]\) is called \textit{bounded} if there exists \( M \in \mathbb{R} \) such that \( |f(x)| \leq M \) for all \( x \in [a, b] \). Negate this, so obtaining the definition of an \textit{unbounded} function.
   
   (b) [8 points] Define what it means to say that “\( a_n \) converges to \( L \)”.
   
   (c) [9 points] Negate your answer in part (b), thus obtaining a definition of “it is false that \( a_n \) converges to \( L \)”. 

NAME:

Total: 200 points. Do 8 out of 12 questions. \textbf{You MUST indicate which 8 questions are to be graded; otherwise, just the first 8 problems will be graded.}
EXPLAIN every answer. No books, notes, calculators or computers allowed on this exam.
Consider a function $f : \mathbb{Z} \to \mathbb{R}$ such that $f(1) = 2$, $f(m) > 0$ for all $m \in \mathbb{Z}$, and

$$f(j - k) = \frac{f(j)}{f(k)} \quad \text{for all } j, k \in \mathbb{Z}.$$ 

Using these properties, find a formula for $f(m), m \in \mathbb{Z}$. (Hint: play around to guess a formula, and then use induction ideas to give a proper proof.)
3 (25 points).
(a) [15 points] Prove that if $n \in \mathbb{N}$ and $q \geq 2$ then $n < q^n$.
(b) [10 points] Prove $\mathbb{N} \times \mathbb{N}$ is countable.
4 (25 points). Prove that with repetition allowed, there are \( \binom{n + k - 1}{k - 1} \) ways to select \( n \) objects from \( k \) types.
5 (25 points). Let $p$ be a prime number.

(a) [12 points] Prove that $p$ divides $\binom{p}{k}$ if $1 \leq k \leq p - 1$. (Hint: count the number of ways to choose a $k$-person subcommittee with a chair, from a $p$-person committee.)

(b) [13 points] Prove that $n^p - n$ is divisible by $p$, for every $n \in \mathbb{N}$. (Hint: try induction, making use of the Binomial Theorem and also part (a) of this question.)
6 (25 points).

(a) [8 points] Prove that if \( a \equiv r \mod n \) and \( b \equiv s \mod n \), then \( a + b \equiv r + s \mod n \) and \( a \cdot b \equiv r \cdot s \mod n \).

(b) [17 points] [Chinese Remainder Theorem] Prove that if \( \{n_i\} \) is a set of \( r \) natural numbers that are pairwise relatively prime, and \( \{a_i\} \) are any \( r \) integers, then the system of congruences \( x \equiv a_i \mod n_i \) has a unique solution modulo \( N = \prod_{i} n_i \).
7 (25 points).

(a) [10 points] Show directly that $(5, 12, 13)$ is a Pythagorean triple, and then show that it has one of the forms

$$(2rs, r^2 - s^2, r^2 + s^2), \quad (r^2 - s^2, 2rs, r^2 + s^2),$$

or possibly a multiple of one of these forms.

(b) [15 points] Fix $c \in \mathbb{Z}$, and define $f(x) = x^6 + cx^5 + 1$. Show that if $c \neq \pm 2$ then $f$ has no rational zeros. Does $f$ have rational zeros when $c = \pm 2$?
8 (25 points).

(b) [8 points] Prove that if \( |b_n - L| \leq a_n \) for all \( n \) and \( a_n \to 0 \), then \( b_n \to L \).

(b) [17 points] Prove that every convergent sequence is a Cauchy sequence.
9 (25 points). Use the definition of limit to prove that \( \lim ((a_n - 4)^{-1}) = \frac{1}{3} \), if \( \lim a_n = 7 \).
10 (25 points).

(a) [8 points] Consider $f$ defined on a deleted neighborhood of $a$. State the definition of what

$$\lim_{x \to a} f(x) = L$$

means.

(b) [8 points] Define what it means for a function $f$ to be continuous at $a$. Define what it means for $f$ to be continuous on the interval $(c, d)$.

(c) [9 points] Give an example of a function $f$ that is defined at every point $x \in [0, 2]$ but that is not continuous at $x = 1$; prove that it is not continuous at $x = 1$. 

11 (25 points). [Intermediate Value Theorem]
Prove that if \( f \) is continuous on \([a, b]\), and \( f(a) < y < f(b) \), then there exists an \( x \in (a, b) \) such that \( f(x) = y \).
12 (25 points). Prove that if $f$ is continuous on $[a, b]$, then $f$ is bounded.