Consider a forced system
\[ x'' + \frac{1}{10} x' + 25x = \cos(\omega t). \]

(a) [0.75 points] Compute a steady periodic (particular) solution.

Solution. Undetermined Coefficients Rule 1 tells us to try
\[ x_p = A \cos \omega t + B \sin \omega t. \]
We do not need Rule 2 because there is no duplication between \( x_c \) and \( x_p \), which you can see because \( x_c \to 0 \) as \( t \to \infty \) (due to the damping), while \( x_p \not\to 0 \). [If you want to be more precise about it, just remember from Section 3.4 that \( x_c \) contains decaying exponential factors, and our guess for \( x_p \) above does not have a decaying exponential factor.]

By substituting
\[ 
\begin{align*}
  x_p &= A \cos \omega t + B \sin \omega t \\
  x'_p &= -\omega A \sin \omega t + \omega B \cos \omega t \\
  x''_p &= -\omega^2 A \cos \omega t - \omega^2 B \sin \omega t
\end{align*}
\]
into the differential equation and then gathering together the cosine and sine terms, we find
\[ (-\omega^2 A + \frac{1}{10} \omega B + 25A) \cos \omega t + (-\omega^2 B - \frac{1}{10} \omega A + 25B) \sin \omega t = \cos \omega t. \]

Equating coefficients of \( \cos \) and \( \sin \) on the two sides of this equation gives us two equations for \( A \) and \( B \):
\[ 
\begin{align*}
  (25 - \omega^2)A + \frac{1}{10} \omega B = 1, \\
  -\frac{1}{10} \omega A + (25 - \omega^2)B = 0.
\end{align*}
\]
The second equation can be solved to give \( A = (25 - \omega^2)B/(\frac{1}{100} \omega) \), and after substituting this into the first equation we find
\[ 
A = \frac{(25 - \omega^2)}{(25 - \omega^2)^2 + (\frac{\omega}{10})^2}, \quad B = \frac{\frac{\omega}{10}}{(25 - \omega^2)^2 + (\frac{\omega}{10})^2}.
\]
Asides.

1. There is no need to expand out the squares, in the formulas for $A$ and $B$.
2. From class we know the solution can be rewritten in the form $x_p = C(\omega) \cos(\omega t - \gamma)$, where the amplitude is

$$C(\omega) = \sqrt{A^2 + B^2} = \frac{1}{\sqrt{(25 - \omega^2)^2 + (\frac{\omega}{10})^2}}.$$ 

(b) [0.25 point] Would the system be close to practical resonance if the forcing frequency were $\omega = 5$? Explain, using whatever information seems relevant.

Solution.

Yes, the system should be close to practical resonance when $\omega = 5$.

Explanation: if the damping term $\frac{1}{10}x'$ were removed from the equation, then the natural frequency would be $\omega_0 = \sqrt{25/1} = 5$ and the system would be in resonance at forcing frequency $\omega = 5$. Since the damping coefficient $c = \frac{1}{10}$ is small compared to the other coefficients in the equation, the system should behave somewhat like in the undamped case, at forcing frequency $\omega = 5$. That is, the system should be close to practical resonance.

Alternatively, you could argue that $C(\omega)$ attains its maximum at precisely $\omega = 4.9995$, which is very close to $\omega = 5$. But the calculations for proving this would take more time than the Quiz allows!