SEC. 9.5 — HEAT EQUATION AND SEPARATION OF VARIABLES

You should learn everything on this handout, and should also learn how to come up with these solutions by using Separation of Variables, as covered in class and in the textbook.

**Theorem 1** (Heat equation, Dirichlet BCs). The solution of

\[ u_t = ku_{xx}, \quad 0 < x < L, \]  

PDE

\[ u(0, t) = u(L, t) = 0, \quad \text{for all } t > 0, \]  

BC

\[ u(x, 0) = f(x) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{L} \right), \]  

IC

is

\[ u(x, t) = \sum_{n=1}^{\infty} b_n \exp \left[ -\left( \frac{n\pi}{L} \right)^2 kt \right] \sin \left( \frac{n\pi x}{L} \right). \]

Note. One can differentiate term-by-term to check \( u_t = ku_{xx} \), and putting \( x = 0 \) gives the boundary condition \( u(0, t) = 0 \) and putting \( x = L \) gives \( u(L, t) = 0 \). Finally, putting \( t = 0 \) gives the initial condition \( u(x, 0) = f(x) \).

**Conclusion.** To solve the heat equation with Dirichlet boundary conditions, first find the sine coefficients \( b_n = \frac{2}{L} \int_0^L f(x) \sin \left( \frac{n\pi x}{L} \right) dx \) of the initial data \( f \), and then write down the series for \( u(x, t) \).

**Observations for Dirichlet BCs**

1. As \( t \to \infty \), \( u(x, t) \to 0 \). ("Dirichlet boundary conditions suck heat out of the rod.")

2. The high modes (the terms with large \( n \)-values) get damped quickly, and so the lowest mode \( (n = 1) \) dominates after a short time, meaning

\[ u(x, t) \approx b_1 \exp \left[ -\left( \frac{\pi}{L} \right)^2 kt \right] \sin \left( \frac{\pi x}{L} \right) \quad \text{when } t \gg 0. \]

**Theorem 2** (Heat equation, Neumann BCs). The solution of

\[ u_t = ku_{xx}, \quad 0 < x < L, \]  

PDE

\[ u_x(0, t) = u_x(L, t) = 0, \quad \text{for all } t > 0, \]  

BC

\[ u(x, 0) = f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi x}{L} \right), \]  

IC

is

\[ u(x, t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \exp \left[ -\left( \frac{n\pi}{L} \right)^2 kt \right] \cos \left( \frac{n\pi x}{L} \right). \]

**Conclusion.** To solve the heat equation with Neumann boundary conditions, first find the cosine coefficients \( a_0 = \frac{2}{L} \int_0^L f(x) dx \) and \( a_n = \frac{2}{L} \int_0^L f(x) \cos \left( \frac{n\pi x}{L} \right) dx \) of the initial data \( f \), and then write down the series for \( u(x, t) \).

**Observations for Neumann BCs**

1. As \( t \to \infty \), \( u(x, t) \to \frac{1}{2} a_0 = \frac{1}{L} \int_0^L f(x) dx = \) (average value of initial temp \( f \)).

   ("Insulated ends don’t let heat escape, so the temp. just averages out over time."))

2. The high modes still damp quickly, and so \( u(x, t) \approx \frac{1}{2} a_0 + a_1 \exp \left[ -\left( \frac{\pi}{L} \right)^2 kt \right] \cos \left( \frac{\pi x}{L} \right). \)