

RESEARCH STATEMENT

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1. INTRODUCTION AND RESEARCH PHILOSOPHY

My primary interests lie in complex analysis in general and CR geometry in particular. My research in CR geometry has also led me to study problems in real and complex algebraic geometry, differential equations, discrete geometry, combinatorics, number theory, and experimental mathematics using computers. My research philosophy is not to simply solve problems within the confines of a particular area, but to look for connections with (and applications to) other areas of mathematics and computer science.

In CR geometry, I am primarily interested in the study of singularities and complexity in CR geometry. When a real object lies in a complex manifold, it inherits a certain amount of the complex structure (called the CR structure) from the ambient manifold. If the object itself is singular, then we may ask about the behaviour of the CR structure as we approach the singularity and about the CR structure of the singularity itself. Even if the real object is not singular, the CR structure may have singularities of its own. Studying these singularities leads to a complexity theory for maps between real objects preserving the CR structure.

More specifically, I study singular Levi-flat hypersurfaces and CR maps between spheres and hyperquadrics. The common link is the equation $\|f(z)\|^2 - \|g(z)\|^2 = 0$ for holomorphic maps f and g . If f and g are scalar-valued the solution set is Levi-flat. When only g is scalar-valued, f/g maps the solution set to the sphere. The study of CR maps between spheres has a rich combinatorial and discrete geometric aspect, which lends itself naturally to computer computation. Many fundamental questions in both areas are not yet fully answered and offer a rich array of topics for future study.

2. CR MAPS BETWEEN SPHERES AND HYPERQUADRICS

Let us start with CR maps of spheres and hyperquadrics. If $M \subset \mathbb{C}^n$ and $M' \subset \mathbb{C}^N$ are real submanifolds, then a map $\varphi: M \rightarrow M'$ is CR if it satisfies the tangential Cauchy-Riemann equations. For example, φ could be the restriction of a holomorphic map. An important question in CR geometry is to classify such maps.

Let M' be a hyperquadric, that is if M' is defined by $\langle z, z \rangle = 1$, where $\langle \cdot, \cdot \rangle$ is a nondegenerate (not necessarily positive definite) Hermitian product. For such M' , the classification of the CR maps $\varphi: M \rightarrow M'$ amounts to understanding the ideal of real functions vanishing on M . For simplicity, we can assume that φ is rational and M is a real algebraic hypersurface. Let $\rho(z, \bar{z}) = 0$ be the defining equation for M , then we can write

$$\rho(z, \bar{z}) = \|f(z)\|^2 - \|g(z)\|^2,$$

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where f and g are holomorphic maps to some finite dimensional space. Interpreted properly, the map (f, g) is a CR map of M to a hyperquadric M' , unique up to fractional linear transformations preserving M' . We can, therefore, understand the CR geometry of M by classifying all the CR maps from M to hyperquadrics.

A well studied case of this problem is when $M \subset \mathbb{C}^n$ and $M' \subset \mathbb{C}^N$ are unit spheres. When $N < n$ no nonconstant CR maps exist. When $n = N = 1$ the map z^d takes the unit circle to the unit circle and is of degree d , which is arbitrary. On the other hand a well-known theorem (by Pincuk, Alexander, and others) states that if $n = N \geq 2$, then any CR map of spheres must be linear fractional (rational map of degree 1).

We will say that two maps are *spherically equivalent* if they are conjugates of each other using automorphisms of the sphere. A map is monomial if each component is a single monomial. The following theorem is proved by using the normal form for a pair Hermitian operators.

Theorem 2.1 (Lebl [11]). *Let $f: S^{2n-1} \rightarrow S^{2N-1}$, $n \geq 2$, be a rational CR map of degree 2. Then f is spherically equivalent to a monomial map.*

In particular, f is equivalent to a map taking $z \in \mathbb{C}^n$ to

$$(1) \quad Lz \oplus (\sqrt{I - L^*L}z) \otimes z,$$

where L is a diagonal matrix with nonnegative diagonal entries sorted by size, such that $I - L^*L$ also has nonnegative entries.

Furthermore, all maps of the form (1) are mutually spherically inequivalent.

By a theorem of Forstnerič [6], if $n \geq 2$, and the map is sufficiently smooth up to the boundary, then the map is rational of degree bounded by a function of n and N only. A sharp bound on the degree is not known. D'Angelo conjectured that the degree d of the map satisfies the sharp bound:

$$(2) \quad d \leq \begin{cases} 2N - 3 & \text{if } n = 2 \\ \frac{N-1}{n-1} & \text{if } n \geq 3. \end{cases}$$

If the bound is true, then it is sharp as there exist monomial maps that achieve equality. When $n = 2$, D'Angelo, Kos, and Riehl [1] proved the bound for monomial maps. It seems reasonable that the combinatorics in the monomial case captures the complexity of the general problem.

Theorem 2.2 (D'Angelo, Lebl, Peters [4]). *Suppose that $f: \mathbb{B}_n \rightarrow \mathbb{B}_N$ is a proper monomial map of degree d , $n \geq 3$. Then*

$$(3) \quad d \leq \frac{4}{3} \frac{N - 3/2}{n - 3/2}, \quad \text{and} \quad d \leq \frac{N - 1}{n - 1} \text{ when } n \gg d.$$

Theorem 2.3 (Lebl, Peters [14]). *Suppose that $f: \mathbb{B}_3 \rightarrow \mathbb{B}_N$ is a proper monomial map of degree d . Then*

$$(4) \quad d \leq \frac{N - 1}{2}.$$

There is also a computational, and thus experimental, aspect of this problem. Many questions, especially about the monomial maps, can be answered computationally. Daniel Lichtblau (Wolfram Research) and I have written independent computer code and ran long computations to classify monomial maps for a fixed

degree, source and target dimensions. In [13] we have, among other results, classified all monomial proper maps from \mathbb{B}_2 to \mathbb{B}_N of the maximal degree $2N - 3$, up to degree $d = 17$. I have extensively used my own mathematics software package Genius [12] to develop and run some of the computations.

3. LEVI-FLAT HYPERSURFACES

Pseudoconvexity is the complex variables analogue of convexity. The natural domain of definition of a holomorphic function has pseudoconvex boundary. An interesting degenerate case is when a real hypersurface is pseudoconvex from both sides. It is then called *Levi-flat*. A Levi-flat hypersurface is locally foliated by complex hypersurfaces.

Suppose that $H \subset \mathbb{C}^N$ is a singular codimension one real analytic subvariety that is Levi-flat near all regular points. We say that H is a Levi-flat *hypervariety*. As H is locally foliated by complex hypersurfaces and it may behave similarly to a complex hypersurface. We can analogously define Levi-flat for higher codimensional manifolds, essentially as an intersection of Levi-flat hypersurfaces in general position. With this definition we also consider complex submanifolds to be Levi-flat. One naturally asks if the singularity of a Levi-flat hypervariety is Levi-flat. I have so far partially answered this question in [8]. I have recently made progress on extending the result for singularities of higher codimension.

Theorem 3.1 (Lebl [8]). *Let $H \subset \mathbb{C}^N$ be a Levi-flat hypervariety. If the singular set H_s is of codimension one in H , and H_s is contained in the closure of regular points of hypersurface type, then H_s is Levi-flat.*

Just as in the case of complex algebraic varieties, it is convenient to study algebraic Levi-flat hypervarieties in complex projective space. In high enough dimension all Levi-flat hypervarieties are singular [15]. Unlike for complex varieties, there is no Chow's theorem; there exist non-algebraic Levi-flat hypersurfaces, see [10]. I did prove the following analogue of Chow's theorem. Known examples suggest that the hypotheses of the theorem cannot be weakened significantly.

Theorem 3.2 (Lebl [10]). *Let $H \subset \mathbb{P}^n$, $n \geq 2$, be an irreducible Levi-flat hypervariety with infinitely many compact leaves, such that locally H is defined by a meromorphic function. Then H is semialgebraic and contained in a pullback of a real-algebraic curve in \mathbb{C} by a rational function.*

It would be useful in studying Levi-flat hypersurfaces to know when we can take a compact codimension-two real submanifold and find a smooth Levi-flat hypersurface with this given boundary. As Levi-flat hypersurfaces are defined by functions satisfying a complex Monge-Ampère type equation, one point of view is to study this question as a boundary value problem for a partial differential equation. In dimension 2 the question has been heavily studied. In dimension 3 and higher, there have been few results (essentially only [5]). See [7] for the full bibliography. I have studied the case of real analytic boundary, both locally and globally. Locally, I have classified [7] all possible real analytic CR submanifolds that are boundaries of a smooth Levi-flat hypersurface. In suitable coordinates (z, w) , they are given by $\{\operatorname{Im} w_1 = f(z, \bar{z}, \bar{w}), \operatorname{Im} w_2 = 0\}$. Unless the boundary itself is Levi-flat, the hypersurface must be real analytic and unique. I have also found the following global regularity and uniqueness result. Note that a generic compact submanifold of codimension 2 has only isolated CR singularities.

Theorem 3.3 (Lebl [9]). *Let $M \subset \mathbb{C}^N$, $N \geq 3$, be a compact real analytic submanifold of codimension 2. Suppose there exists a compact connected C^∞ Levi-flat hypersurface H with boundary, such that $\partial H = M$.*

If the CR singularities of M are isolated, then $H \setminus M$ is real analytic. Further, H is the unique compact connected Levi-flat C^∞ hypersurface with boundary M .

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