

Math 518 Differentiable Manifolds I

Assignment 7, Due Tuesday December 8

- (a) For a vector field X on a manifold M and a k -form ω on M , deduce what the definition of $\mathcal{L}_X\omega$, the Lie derivative of ω with respect to X , should be.
(b) For a vector field X on M one can define the contraction mapping $i_X: \Omega^k(M) \rightarrow \Omega^{k-1}(M)$ by

$$i_X\omega(V_1, \dots, V_{k-1}) = \omega(X, V_1, \dots, V_{k-1}).$$

This is related to the Lie derivative by **Cartan's magic formula**:

$$\mathcal{L}_X\omega = d(i_X\omega) + i_X(d\omega).$$

See if you can prove this identity. (Hint: start with the right hand side and the algebraic definition of the exterior derivative.)

- The classical Green's formula states that if $W \subset \mathbb{R}^2$ is a domain with a smooth boundary curve γ , then for any two functions $f, g: \mathbb{R}^2 \rightarrow \mathbb{R}$ we have

$$\int_{\gamma} f dx + g dy = \int_W \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx dy.$$

Prove that this follows from our generalized version of Stokes Theorem.

- Let M be a compact oriented manifold of dimension m with nonempty boundary. Let $F: \partial M \rightarrow N$ be a smooth map to a manifold N of dimension $m - 1$ and let ω be a smooth $m - 1$ -form on N . Prove that if F extends to all of M then $\int_{\partial M} F^*\omega = 0$.
- Let $F_0, F_1: M \rightarrow N$ be homotopic maps, where M is a compact oriented manifold without boundary and M and N both have dimension m . Prove that for all m -forms ω on N we have

$$\int_M F_0^*\omega = \int_M F_1^*\omega.$$

Hint: previous question.

5. Let S^2 denote the unit sphere in \mathbb{R}^3 defined by

$$x^2 + y^2 + z^2 = 1$$

given the standard orientation. That is, $\{\partial/\partial y, \partial/\partial z\}$ is positively oriented at $(1, 0, 0)$. Let α be the 2-form defined by restricting

$$\alpha = (3x^2 \cos(y) + e^{xy}) dx \wedge dy + 17x^3 dx \wedge dz + (x + yz^5 + \sin(z)) dy \wedge dz$$

to S^2 . Compute

$$\int_{S^2} \alpha.$$