Math 518 Differentiable Manifolds I

Assignment 6, Due Thursday November 19

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1. Let A be a linear map of an m-dimensional vector space V to itself, and let $\omega \in \Lambda^m(V^*)$. Compute $A^*\omega$.

2. A non-zero k-form $\phi \in \Lambda^k(V^*)$ is called *decomposable* if $\phi = \phi_1 \wedge \ldots \wedge \phi_k$ where the ϕ_j are all 1-forms. Show that if $\dim(V) < 4$ then every non-zero element of $\Lambda^2(V^*)$ is decomposable. Give a counterexample in dimension four.

3. Let $h\colon \mathbb{R}^1 \to S^1$ be $h(t) = (\cos t, \sin t)$. Show that if ω is any 1-form on S^1, then
$$\int_{S^1} \omega = \int_0^{2\pi} h^*\omega.$$

4. A *closed curve* on a manifold M is a map $\gamma\colon S^1 \to M$. If ω is a 1-form on M, define the *line integral* of ω around γ by
$$\oint_\gamma \omega = \int_{S^1} \gamma^*\omega.$$
If $M = \mathbb{R}^k$, write $\oint_\gamma \omega$ explicitly in the coordinate expressions of γ and ω.

5. Prove that a 1-form ω on S^1 is the differential of a function iff $\int_{S^1} \omega = 0$. [Hint: "Only if" follows from Q4. Now let h be as in Q3, and define a function g on $\mathbb{R}$ by
$$g(t) = \int_0^t h^*\omega.$$
Show that if $\int_{S^1} \omega = 0$ then $g(t + 2\pi) = g(t)$. Therefore $g = f \circ h$ for some function f on S^1. Check $df = \omega$.]

6. Let ν be any 1-form in S^1 with non-zero integral. Prove that if ω is any other 1-form, then there is a constant c such that $\omega - c\nu = df$ for some function f on S^1.