HW 5 SOLUTIONS, MA518

1. Problem 1

The sphere S^{2n-1} is the set of vectors in $\mathbb{C}^n$ with unit norm i.e. vectors $\mathbf{z} = (z_1, \cdots, z_n)$ such that $|\mathbf{z}|^2 = | z_1 |^2 + \cdots + | z_n |^2$. Consider the parameterized family of maps $f_t : S^{2n-1} \to S^{2n-1}$ defined by

$$f_t(\mathbf{z}) = e^{it\pi}\mathbf{z}$$

Then f_t is a homotopy from the identity map to the antipodal map. The hint given in the problem is exactly the same idea: multiplication by $e^{it\pi}$ on $\mathbb{C}$ as a real linear map (forgetting the complex structure) is the same as the action on $\mathbb{R}^2$ of the linear map in the hint.

2. Problem 2

For the (complex) vector $\mathbf{w} = (w_1, \cdots, w_n)$ in S^{2n-1}, consider the map $F(t) = df_t(\mathbf{w})$ from Problem 1. The derivative dF satisfies

$$dF\left(\frac{\partial}{\partial t}\right)_{t=0} = \pi i \sum_{i=1}^{n} w_i \frac{\partial}{\partial z_i}$$

The vector field on the right hand side above is a non-vanishing vector field on S^{2n-1}.

3. Problem 4

Let F be the map $\mathbb{R}^l \times S \to \mathbb{R}^n$ in the hint. It is straightforward to check that F is a submersion. The tangent space at a point in $\mathbb{R}^l \times S$ is the product $T_t\mathbb{R}^l \times (T_v\mathbb{R}^n)^l$. So for instance, if at the point we are considering $t_1 \neq 0$, then

$$dF\left(0, (\frac{\partial}{\partial x_1}, 0, \cdots, 0)\right) = t_1 \frac{\partial}{\partial x_1}$$

which indicates why F is a submersion.

This implies that F is transverse to M. By the Transversality Theorem, for almost every $s \in S$, the map $f_s : \mathbb{R}^l \to \mathbb{R}^n$ given by $f_s(\mathbf{t}) = F(\mathbf{t}, s)$ is transverse to M. Hence, almost every l-dimensional subspace intersects M transversally.

4. Problem 5

Since M is connected, the point x can be joined to y by a smooth simple (does not self intersect) path g. Think of g as a smooth map from I to M such that $g(0) = x$ and $g(1) = y$. Push-forward the vector field $\partial/\partial t$ on I by the derivative dg to get the tangent vector field $dg(\partial/\partial t)$ to the path. Extend this tangent vector field to a smooth vector field X with compact support (use bump functions to achieve compact support). The time 1 flow of X (which exists because of compact support) is then a diffeomorphism of M that sends x to y.

5. Problem 7

$$
\begin{aligned}
XY &= \left(x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} - zx\frac{\partial}{\partial z}\right)\left(y^2\frac{\partial}{\partial x} - xyz\frac{\partial}{\partial z}\right) \\
&= \left(x^2(-yz)\frac{\partial}{\partial z}\right) + \left(xy(2y)\frac{\partial}{\partial x} + xy(-xz)\frac{\partial}{\partial z}\right) + \left(-zx(-xy)\frac{\partial}{\partial z}\right) \\
&= 2xy^2\frac{\partial}{\partial x} - x^2yz\frac{\partial}{\partial z}
\end{aligned}
$$

Similarly

$$
\begin{aligned}
YX &= \left(y^2\frac{\partial}{\partial x} - xyz\frac{\partial}{\partial z}\right)\left(x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} - zx\frac{\partial}{\partial z}\right) \\
&= \left(y^2(2x)\frac{\partial}{\partial x} + y^2(y)\frac{\partial}{\partial y} - y^2(z)\frac{\partial}{\partial z}\right) + \left((-xyz)(-x)\frac{\partial}{\partial z}\right) \\
&= 2xy^2\frac{\partial}{\partial x} + y^3\frac{\partial}{\partial y} + (x^2yz - y^2z)\frac{\partial}{\partial z}
\end{aligned}
$$

Hence the Lie bracket is

$$[X,Y] = (y^2z - 2x^2yz)\frac{\partial}{\partial z} - y^3\frac{\partial}{\partial y}$$