Math 518 Differential Manifolds I

Assignment 5, Due Thursday October 22

---

1. Show that the antipodal map $x \to -x$ of S^n is homotopic to the identity if n is odd. Hint: Start with $n = 1$ by using the linear maps
$$\begin{pmatrix} \cos \pi t & -\sin \pi t \\ \sin \pi t & \cos \pi t \end{pmatrix}.$$

2. Construct a nonvanishing vector field on S^n for n odd.

3. Construct a vector field on S^2 which vanishes at precisely one point. Hint: Consider S^2 as the Riemann sphere $\mathbb{C} \cup \infty$ and write down the flow of your vector field.

4. Suppose that M is a submanifold of $\mathbb{R}^n$. Show that "almost every" vector subspace V of $\mathbb{R}^n$ with fixed dimension l, intersects M transversally. Hint: The subset $S \subset (\mathbb{R}^n)^l$ consisting of all linear independent l-tuples of vectors in $\mathbb{R}^n$ is open, and the map $\mathbb{R}^l \times S \to \mathbb{R}^n$ defined by
$$[(t_1, \ldots, t_l), v_1, \ldots, v_l] \mapsto t_1 v_1 + \cdots + t_l v_l$$
is a submersion.

5. Prove that for every connected manifold M and any pair of points x,$y \in M$ there is a diffeomorphism ϕ of M which takes x to y. Hint: constuct ϕ using the flow of a vector field.

6. If U is an open subset of M prove that $T_x U = T_x M$ for all $x \in U$.

7. Consider the following two vector fields on $\mathbb{R}^3$
$$X(x, y, z) = (x^2, xy, -zx) \quad Y(x, y, z) = (y^2, 0, -xyz).$$
Compute $[X, Y]$.

8. Let X, Y, and Z be smooth vector fields on M. Verify the **Jacobi Identity**:
$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$