On $r$-dynamic Coloring of Graphs

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Abstract

An $r$-dynamic proper $k$-coloring of a graph $G$ is a proper $k$-coloring of $G$ such that every vertex in $V(G)$ has neighbors in at least $\min\{d(v), r\}$ different color classes. The $r$-dynamic chromatic number of a graph $G$, written $\chi_r(G)$, is the least $k$ such that $G$ has such a coloring. By a greedy coloring algorithm, $\chi_r(G) \leq r\Delta(G) + 1$; we prove that equality holds for $\Delta(G) > 2$ if and only if $G$ is $r$-regular with diameter 2 and girth 5. We improve the bound to $\chi_r(G) \leq \Delta(G) + 2r - 2$ when $\delta(G) > 2r \ln n$ and $\chi_r(G) \leq \Delta(G) + r$ when $\delta(G) > r^2 \ln n$.

In terms of the chromatic number, we prove $\chi_r(G) \leq r\chi(G)$ when $G$ is $k$-regular with $k \geq (3 + o(1))r \ln r$ and show that $\chi_r(G)$ may exceed $r^{1.377} \chi(G)$ when $k = r$. We prove $\chi_2(G) \leq \chi(G) + 2$ when $G$ has diameter 2, with equality only for complete bipartite graphs and the 5-cycle. Also, $\chi_2(G) \leq 3\chi(G)$ when $G$ has diameter 3, which is sharp. However, $\chi_2$ is unbounded on bipartite graphs with diameter 4, and $\chi_3$ is unbounded on bipartite graphs with diameter 3 or 3-colorable graphs with diameter 2. Finally, we study $\chi_r$ on cartesian products of two paths or two cycles.

1 Introduction

In a communication network, adjacent computers must be given distinct resources. To improve accessibility of resources, each computer should be able to find many of the resources among its neighbors. Requiring all neighbors of each computer to have different resources requires too many types of resources. Instead, we specify a threshold $r$; the neighbors of a

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Graph coloring provides a model. A \( k \)-coloring of a graph \( G \) is a map \( c : V(G) \rightarrow S \), where \( |S| = k \). A coloring is proper if adjacent vertices receive different labels. An \( r \)-dynamic \( k \)-coloring is a proper \( k \)-coloring \( c \) of \( G \) such that \( |c(N(v))| \geq \min\{r, d(v)\} \) for each vertex \( v \) in \( V(G) \), where \( N(v) \) is the neighborhood of \( v \) and \( c(U) = \{c(v) : v \in U\} \) for a vertex subset \( U \). The \( r \)-dynamic chromatic number, introduced by Montgomery [14] and written as \( \chi_r(G) \), is the minimum \( k \) such that \( G \) has an \( r \)-dynamic \( k \)-coloring.

The 1-dynamic chromatic number of a graph \( G \) is its chromatic number \( \chi(G) \). The 2-dynamic chromatic number was introduced as dynamic chromatic number by Montgomery [14]; he conjectured \( \chi_2(G) \leq \chi(G) + 2 \) when \( G \) is regular, which remains open. Alishahi [4] showed that for all \( k \) there is a \( k \)-chromatic regular graph \( G \) with \( \chi_2(G) \geq \chi(G) + 1 \). Akbari et al. [1] proved Montgomery’s conjecture for bipartite regular graphs. Lai, Montgomery, and Poon [11] proved \( \chi_2(G) \leq \Delta(G) + 1 \) when \( \Delta(G) \geq 3 \) and no component is the 5-cycle \( C_5 \).

Akbari et al. [2] strengthened this to the list context: \( \chi_d(G) \leq \Delta(G) + 1 \) under the same conditions, where \( \chi_d(G) \) is the least \( k \) such that a 2-dynamic coloring can be chosen from any lists of size \( k \) assigned to the vertices. Kim and Park [9] proved \( \chi_d(G) \leq 4 \) for planar \( G \) with girth at least 7, and \( \chi_d(G) \leq k \) when \( k \geq 4 \) and \( G \) has maximum average degree at most \( \frac{4k}{k+2} \) (both results are sharp). Kim, Lee, and Park [10] proved \( \chi_2(G) \leq 4 \) when \( G \) is planar and no component is \( C_5 \); also, \( \chi_d(G) \leq 5 \) whenever \( G \) is planar.

Given a graph \( G \), let \( G^2 \) denote the graph obtained from \( G \) by adding edges joining nonadjacent vertices that have a common neighbor. Properly coloring \( G^2 \) is a restricted version of properly coloring \( G \); require vertices with the same color to be distance at least 3 apart. One motivation for the study of \( r \)-dynamic chromatic number is that it provides a spectrum of parameters between \( \chi(G) \) and \( \chi(G^2) \).

**Observation 1.1.** Always \( \chi(G) = \chi_1(G) \leq \cdots \leq \chi_{\Delta(G)}(G) = \chi(G^2) \). If \( r \geq \Delta(G) \), then \( \chi_r(G) = \chi_{\Delta(G)}(G) \).

**Observation 1.2.** \( \chi_r(G) \geq \min\{\Delta(G), r\} + 1 \), and this is sharp.

**Proof.** The closed neighborhood of a vertex of maximum degree needs \( \min\{\Delta(G), r\} + 1 \) colors. When \( G \) is a tree, take one vertex as a root and iteratively color children of each vertex greedily with colors different from the parent; \( \min\{\Delta(G), r\} + 1 \) colors suffice. Equality also holds for cycles whose length is divisible by 6, for example.

We begin in Section 2 with an analogue of Brooks’ Theorem [7]: we prove \( \chi_r(G) \leq r\Delta(G) + 1 \) and characterize the graphs achieving equality. Our result is not a generalization of Brooks’ Theorem, because although \( \chi_1(G) = \chi(G) \), the characterization of equality for \( r \geq 2 \) does not reduce to Brooks’ Theorem when \( r = 1 \). For \( r \geq 2 \), equality holds if and only if \( G \) is an \( r \)-regular graph with diameter 2 and girth 5. Such graphs, known as Moore
graphs, are quite rare; the only examples are $C_5$, the 3-regular Petersen graph, the 7-regular Hoffman–Singleton graph, and possibly a 57-regular graph not known to exist. (The diameter of a graph $G$, written $\text{diam}(G)$, is the maximum of the distances between vertices.)

When the minimum degree is not too small, we can greatly improve the bound in terms of the maximum degree. For an $n$-vertex graph $G$, we show that $\delta(G) > \frac{r^2}{r+1} r \ln n$ implies $\chi_r(G) \leq \Delta(G) + r + s$. In particular, $\delta(G) > 2r \ln n$ implies $\chi_r(G) \leq \Delta(G) + 2r$ (setting $s = r - 2$), and $\delta(G) > r^2 \ln n$ implies $\chi_r(G) \leq \Delta(G) + r$ (setting $s = 0$).

In Section 3, we study bounds for regular graphs in terms of the chromatic number, motivated by Montgomery’s conjecture. Akbari et al. [1] proved $\chi_2(G) \leq 2\chi(G)$ for every $k$-regular graph $G$. Alishahi [3] proved $\chi_2(G) \leq \chi(G) + 14.06 \ln k + 1$ (and later $\chi_2(G) \leq \chi(G) + 2 \left[ 4 \ln k + 1 \right]$ [4]), which Taherkhani [16] improved to $\chi_2(G) \leq \chi(G) + \left[ 5.437 \ln k + 2.721 \right]$. For general $r$, we prove $\chi_r(G) \leq r\chi(G)$ for $r$-regular graphs $G$ with $k \geq (3 + o(1)) r \ln r$. The thesis of the third author [15] contains the same conclusion when $k \geq 7r \ln r$, and later Taherkhani [16] obtained a similar result by essentially the same method as ours. When $k$ is not sufficiently large in terms of $r$, the ratio $\chi_r(G) / \chi(G)$ can grow superlinearly in $r$; we provide an example using Kneser graphs where $k = r$ and $\chi_r(G) > r^{1.37744} \chi(G)$.

In Section 4 we consider $k$-chromatic graphs with small diameter. We first give a short proof of $\chi_2(G) \leq \chi(G) + 2$ for every graph $G$ with diameter 2. Furthermore, equality holds only for complete bipartite graphs and $C_5$. For regular graphs with $\chi(G) \geq 4$, Alishahi [4] proved $\chi_2(G) \leq \chi(G) + \alpha(G)^2$ (here $\alpha(G)$ is the maximum size of an independent set of vertices and $G^2$ is obtained from $G$ by adding edges joining vertices with common neighbors; note that $\alpha(G^2) = 1$ when $\text{diam}(G) = 2$).

Moving on to diameter 3, we prove $\chi_2(G) \leq 3\chi(G)$ when $\text{diam}(G) \leq 3$, and this is sharp. For graphs with diameter 4 there is no bound in terms of chromatic number: subdividing every edge of the complete graph $K_n$ yields a bipartite graph $G$ with $\text{diam}(G) = 4$ and $\chi_2(G) = n$. In contrast, $\chi_3$ is unbounded already on bipartite graphs with diameter 2.

To study $\chi_2$ on graphs with diameter 3 we use a related notion from hypergraph coloring. A vertex coloring of a hypergraph $H$ is $c$-strong if every edge $e$ has at least $\min\{c, |e|\}$ distinct colors; this concept was introduced in [5]. Let $\chi(H; c)$ denote the minimum number of colors in a $c$-strong coloring of $H$. Like $\chi_r$, this concept yields a spectrum of parameters, from ordinary hypergraph coloring to “strong coloring”, where all vertices in an edge have distinct colors. A coloring of a graph $G$ is $r$-dynamic if it is proper and is an $r$-strong coloring of the hypergraph $H$ with vertex set $V(G)$ whose edges are the vertex neighborhoods in $G$. Thus $\chi_r(G) \geq \chi(H; r)$. More importantly, $\chi_r(G) \leq \chi(G) \chi(H; r)$, by combining a proper coloring of $G$ with an $r$-strong coloring of $H$.

Finally, in Section 5 we study the behavior of $\chi_r$ under cartesian products. The cartesian product of $G$ and $H$, written $G \square H$, is the graph with vertex set $V(G) \times V(H)$ such that $(u, v)$ and $(u', v')$ are adjacent if and only if the pairs are equal in one coordinate and adjacent in the other. An easy general upper bound when $\delta(G) \geq r$ is $\chi_r(G \square H) \leq \max\{\chi_r(G), \chi(H)\}$.
We also determine $\chi_r(P_m \square P_n)$ in most cases, where $P_q$ denotes the path with $q$ vertices. For simplicity, consider only $m, n \geq 3$. When $r \geq 4$, always $\chi_4(P_m \square P_n) = 5$. For $r = 3$, we have $\chi_3(P_m \square P_n) = 4$ if $m$ and $n$ are both even, and we show that $\chi_3(P_m \square P_n) = 5$ otherwise except possibly when $mn \equiv 2 \mod 4$. The proof that $\chi_3(P_m \square P_n) = 5$ also in that case has been completed by Kang, Müller, and West [8]. For products of cycles we provide some partial results.

2 Bounds in Terms of Maximum Degree

We prove an upper bound on $\chi_r(G)$ in terms of $\Delta(G)$ and characterize (for $r \geq 2$) when equality holds. Not only is the result analogous to Brooks’ Theorem, also the proof idea is like that of a well-known proof of Brooks’ Theorem by Lovász [12].

**Theorem 2.1.** $\chi_r(G) \leq r\Delta(G) + 1$, with equality for $r \geq 2$ if and only if $G$ is $r$-regular with diameter 2 and girth 5.

**Proof.** If $G$ is $r$-regular, then vertices having a common neighbor must receive distinct colors in an $r$-dynamic coloring. If diam$(G) = 2$, then all pairs of vertices are adjacent or have a common neighbor. Hence $\chi_r(G) = |V(G)|$. The maximum value of $|V(G)|$ is $r^2 + 1$, which occurs if and only if $G$ has girth 5. Hence equality holds for Moore graphs.

For the upper bound, we may assume that $G$ is connected, by considering each component. Choose a vertex $v_n$, and use a spanning tree to order the vertices as $v_1, \ldots, v_n$ so that each vertex before $v_n$ has a higher-indexed neighbor; this is an ascending ordering to $v_n$. Color the vertices in the order $v_1, \ldots, v_n$. A vertex is dangerous when its neighborhood does not yet have $r$ colors. When coloring $v_i$, avoid each color used on a neighbor of $v_i$ or on a neighbor of a dangerous neighbor of $v_i$. This is the greedy coloring procedure with respect to the ordering; the first $\min\{d(v), r\}$ colors assigned to the neighborhood of any vertex $v$ are distinct. At most $r$ colors must be avoided for each neighbor of $v_i$.

For $i < n$, the uncolored higher-indexed neighbor of $v_i$ means that at most $r\Delta(G) - 1$ colors need to be avoided when coloring $v_i$. Hence $r\Delta(G)$ colors suffice, except possibly for the last vertex. When $G$ is not regular, choosing $v_n$ to be a vertex of minimum degree completes the proof, since $v_n$ needs to avoid at most $r(\Delta(G) - 1)$ colors.

Hence we may assume that $G$ is regular. If $\Delta(G) > r$, then when $v_n$ is colored in an ascending ordering it is the last uncolored vertex. Under the greedy procedure, its neighbors already have at least $r$ distinct colors in their neighborhoods, so none are dangerous. Therefore only $\Delta(G)$ colors need to be avoided in coloring $v_n$. If $\Delta(G) < r$, then Observation 1.1 yields $\chi_r(G) = \chi(G^2) \leq \Delta(G^2) + 1 \leq \Delta(G)^2 + 1 \leq r\Delta(G)$.

Hence we may assume that $G$ is $r$-regular. If $G$ has a cycle of length at most 4, then let $v_n$ be a vertex on a shortest cycle $C$ and color greedily in an ascending ordering to $v_n$. At
most $r^2$ colors are used in coloring $v_1, \ldots, v_{n-1}$. When coloring $v_n$, the two neighbors of $v_n$ on $C$ generate at least one common color to be avoided (on a vertex of $C$), leaving at most $r^2 - 1$ colors to be avoided. Thus $r^2$ colors suffice.

Hence we may assume that $G$ has girth at least 5, which yields $r^2$ other vertices within distance 2 of each vertex. If $\text{diam}(G) = 2$, then we are finished, so assume $\text{diam}(G) \geq 3$. Let $u$ and $w$ be vertices at distance 3, with $\langle u, x, y, w \rangle$ an induced path. Let $T_1, \ldots, T_k$ be the components of $G - \{u, w\}$, with $x, y \in T_k$.

Color $u$ and $w$ first (each get color 1), and then use an increasing ordering in each $T_i$ to a neighbor of $u$ or $w$, leaving $T_k$ last with an increasing ordering to $x$. As usual, at most $r^2 - 1$ colors must be avoided on any vertex of $T_i$ before the last. For $i < k$, the last vertex $v$ in $T_i$ has an uncolored vertex at distance 2 (it is $x$ or $y$), so it needs to avoid at most $r^2 - 1$ colors. Finally, when coloring $x$, the two vertices $u$ and $w$ have the same color, so again at most $r^2 - 1$ colors need to be avoided on $x$.

We believe that also there is no graph $G$ with $\chi_r(G) = r\Delta(G)$ other than cycles whose length is not divisible by 3 (when $r = 2$); that is, when $\Delta(G) > 2$ and Moore graphs are excluded, the upper bound should improve further. It is known for example that $\chi_2(G) \leq \Delta(G) + 1$ when $\Delta(G) \geq 3$ and no component of $G$ is $C_5$ [11]. We present a restricted construction where the bound cannot be improved by much.

**Example 2.2.** Graphs with $\chi_r(G) = r\Delta(G) - 1$ when $r = \Delta(G)$. When $r = \Delta(G)$, deleting an edge $uv$ from a Moore graph with $\Delta(G) > 2$ yields a graph $G$ with $\chi_r(G) = r\Delta(G) - 1 = r^2 - 1$. For the lower bound, any two vertices in $V(G) - \{u, v\}$ are adjacent or have a common neighbor, so they must have distinct colors. For the upper bound, give distinct colors to $V(G) - \{u, v\}$, give $u$ a color in $N(v)$, and give $v$ a color in $N(u)$; now no color is on two vertices with a common neighbor in $G$.

Let $G_i$ for $i \in \mathbb{Z}_k$ be a copy of this graph $G$. Add the edges joining the copy of $v$ in $G_i$ to the copy of $u$ in $G_{i+1}$, for all $i \in \mathbb{Z}_k$. Since $\Delta(G) > 2$, the duplicated colors in successive copies of $G$ can be chosen to be distinct. Thus infinitely many 2-connected graphs are constructed with $r$-dynamic chromatic number $r\Delta(G) - 1$, but only for $\Delta(G) = r$ and $r \in \{2, 3, 7\}$ (and possibly 57).

**Question 2.3.** For fixed $r$ and $k$, what is the best bound on $\chi_r(G)$ that holds for all but finitely many graphs $G$ with maximum degree $k$?

We show next that if the minimum degree is not too small relative to the number of vertices (for fixed $r$), then the bound in Theorem 2.1 can be improved by replacing the product with a sum involving $\Delta(G)$ and $r$. The idea is to modify the greedy coloring algorithm used in Theorem 2.1. There we ensured that the first $r$ neighbors of a vertex would have distinct colors. Now we will allow $r$ distinct colors to be obtained at any time.
Theorem 2.4. If \( G \) is an \( n \)-vertex graph, and \( \delta(G) > \frac{r - 1}{r + 1} r \ln n \), then \( \chi_r(G) \leq \Delta(G) + r + s \). In particular, \( \delta(G) > 2r \ln n \) implies \( \chi_r(G) \leq \Delta(G) + 2r \), and \( \delta(G) > r^2 \ln n \) implies \( \chi_r(G) \leq \Delta(G) + r \).

Proof. The special cases arise by setting \( s = r - 2 \) and \( s = 0 \) in the general statement.

If \( n \leq \Delta(G) + r + s \), then we can give the vertices distinct colors, so we may assume
\( n > \Delta(G) + r + s \). Let \( v_1, \ldots, v_n \) be any vertex ordering of \( G \). Color \( v_1, \ldots, v_n \) in order using \( \Delta(G) + r + s \) colors. Give \( v_i \) a color chosen uniformly at random among those not used on its earlier neighbors; at least \( r + s \) colors are available. This produces a proper coloring.

We claim that with positive probability the coloring is also \( r \)-dynamic. This fails at a vertex \( v \) only if the colors in \( N(v) \) are confined to a particular set of \( r - 1 \) colors. The probability that this happens with a particular set of \( r - 1 \) colors is bounded by \( \frac{r}{r + s} \), which in turn is bounded by \( e^{-\delta(G) \frac{r}{r + s}} \). There are \( \left( \frac{\Delta(G) + r + s}{r - 1} \right) \) choices of a set of \( r - 1 \) colors, which is less than \( n^{r-1} \) since \( \Delta(G) + r + s < n \).

Since \( G \) has \( n \) vertices, the probability of having a bad vertex is less than \( n^r e^{-\delta(G) \frac{r}{r + s}} \). The constraint on \( \delta(G) \) bounds this by \( n^r n^{-r} \), which equals 1.

3 Regular Graphs and Chromatic Number

To strengthen \( \chi_r(G) \leq r\Delta(G) \) for non-Moore graphs, we want to replace \( \Delta(G) \) with a value no larger. In general, \( r\chi(G) \) would be a better upper bound, since \( \chi(G) \leq \Delta(G) \) by Brooks' Theorem when \( \Delta(G) \geq 3 \) and \( G \) is not complete. We prove \( \chi_r(G) \leq r\chi(G) \) for regular graphs with sufficiently large degree in terms of \( r \). The Petersen graph shows that the inequality does not hold for all \( G \) when \( r = 3 \).

For \( k \)-regular graphs with \( k \) sufficiently large in terms of \( r \), we use a random \( r \)-coloring of the vertices to show that some coloring puts \( r \) distinct colors into each vertex neighborhood. We then assign each vertex a pair consisting of its color under this \( r \)-coloring and its color under a proper \( \chi(G) \)-coloring. This ensures that adjacent vertices have distinct colors; in total, \( r\chi(G) \) color pairs are used.

Lemma 3.1. Let \( H \) be a hypergraph in which each edge has size at least \( k \) and each vertex appears in at most \( D \) edges. If \( ep(k(D - 1) + 1) \leq 1 \), where \( p = re^{-k/r} \), then some \( r \)-coloring of \( V(H) \) puts all \( r \) colors into each edge of \( H \).

Proof. Color the vertices of \( H \) independently and uniformly at random from a set of \( r \) colors. For \( e \in E(H) \), let \( A_e \) be the event that some color fails to appear on \( e \). Note that \( \Pr(A_e) \leq r(1 - 1/r)^k \leq re^{-k/r} \). The event \( A_e \) is determined by choices on the vertices of \( e \), so \( A_e \) is mutually independent of the set of all events for edges that do not intersect \( e \). These include all but at most \( k(D - 1) + 1 \) events. By the Local Lemma, \( ep(k(D - 1) + 1) \leq 1 \).
guarantees that in some outcome of the random coloring none of the edge events occurs, meaning that all \( r \) colors appear on each edge of \( H \).

\[ \text{Theorem 3.2.} \text{ If } G \text{ is a } k\text{-regular graph and } ep(k(k-1)+1) \leq 1, \text{ where } p = re^{-k/r}, \text{ then } \chi_r(G) \leq r\chi(G). \text{ In particular, if } k \geq (3+x)r\ln r, \text{ where } x = \frac{2\ln\ln r}{\ln r} \text{ is a small positive constant, then } \chi_r(G) \leq r\chi(G). \]

\[ \text{Proof.} \text{ Define a hypergraph } H \text{ by } V(H) = V(G) \text{ and } E(H) = \{N(v): v \in V(G)\}. \text{ Note that } H \text{ is } k\text{-uniform and that every vertex in } H \text{ appears in } k \text{ edges. By Lemma 3.1, since } ep(k(k-1)+1) \leq 1 \text{ by hypothesis, there is an } r\text{-coloring of } V(H) \text{ such that every edge in } E(H) \text{ has } r \text{ colors on it. As observed before Lemma 3.1, pairing this coloring with a proper coloring of } G \text{ yields an } r\text{-dynamic coloring of } H \text{ with } r\chi(G) \text{ colors.}

\text{The second claim holds because } k \geq (3+x)r\ln r \text{ with } x > \frac{2\ln\ln r}{\ln r} \text{ implies } ere^{-k/r}k^2 \leq 1, \text{ which is stronger than the needed inequality for } k. \]

Taherkhani [16] independently discovered essentially the same argument as Lemma 3.1. He did not simplify the resulting threshold on \( k \) in terms of \( r \) as in Theorem 3.2. In the thesis of the third author [15], the threshold \( k \geq 7r\ln r \) was given by a similar method. We do not know the least \( k \) to guarantee \( \chi_r(G) \leq r\chi(G) \) when \( G \) is \( k \)-regular, but we can show that when \( k = r \) the ratio \( \chi_r(G)/\chi(G) \) can grow superlinearly in \( r \). Let \([n]\) denote \( \{1, \ldots, n\} \).

\[ \text{Theorem 3.3.} \text{ For infinitely many } r, \text{ there is an } r\text{-regular graph } G \text{ such that } \chi_r(G) > r^{1.37744}\chi(G). \]

\[ \text{Proof.} \text{ The Kneser graph } K(n,t) \text{ is the graph whose vertices are the } t\text{-element subsets of } [n], \text{ with two sets adjacent when they are disjoint. Each vertex is adjacent to } \binom{n-t}{t} \text{ other vertices. Given } t \in \mathbb{N}, \text{ let } G = K(3t-1,t) \text{ and } r = \binom{n-t}{t}. \text{ Any two nonadjacent vertices in } G \text{ have a common neighbor, since two intersecting } t\text{-sets in } [3t-1] \text{ omit at least } t \text{ elements, so } \text{diam}(G) = 2. \text{ Since } G \text{ is } r\text{-regular, we thus have } \chi_r(G) = |V(G)| = \binom{3t-1}{t}. \text{ On the other hand, Lovász [13] and Bárány [6] proved } \chi(K(n-t)) = n-2t+2, \text{ so } \chi(G) = t+1.

\text{It remains to express } \chi_r(G) \text{ in terms of } r \text{ and } \chi(G). \text{ In terms of } t, \text{ we have } r = \binom{2t-1}{t} = \frac{1}{2}\left(\frac{3t}{2}\right) \text{ and } \chi_r(G) = \binom{3t-1}{t} = \frac{2}{3}\left(\frac{3t}{2}\right). \text{ Note that } \chi_r(G) \text{ and } r \text{ are both very much larger than } \chi(G), \text{ so what is important is the ratio between } \chi_r(G) \text{ and } r \text{ as a function of } r. \text{ For } c \in \{\frac{1}{2}, \frac{1}{3}\}, \text{ we use the approximation}

\[ \left(\frac{m}{cm}\right) \approx \frac{(c^t(1-c)^{1-c})^{-m}}{c(1-c)^{2\pi m}}, \]

that arises from Stirling’s Formula \( m! = m^me^{-m}\sqrt{2\pi m}. \text{ We compute}

\[ \frac{\chi_r(G)}{r\chi(G)} \approx \frac{\frac{2}{3}\left(\frac{3t}{2}\right)}{(t+1)\frac{2}{3}\left(\frac{3t}{2}\right)} \approx \frac{4}{3t4^t\sqrt{(4/3)\pi t}} \approx \frac{1}{t}\sqrt{\frac{4}{3}}\left(\frac{27}{16}\right)^t. \]

\text{Setting this ratio to be } r^x, \text{ where } r \approx \frac{1}{4}\sqrt[4]{\frac{3}{\pi t}}, \text{ we take logarithms to obtain } t\lg(27/16) = (1+o(1))tx\lg 4, \text{ which simplifies to } x = \frac{1}{2}(\lg 27 - 4) > .37744. \text{ Thus } \chi_r(G) > r^{1.37744}\chi(G). \]

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Theorems 3.2 and 3.3 suggest the following question.

**Question 3.4.** As a function of \( r \), what is the least \( k \) such that \( \frac{\chi(G)}{\chi(G)} \) is at most linear in \( r \) when \( G \) is \( k \)-regular? What is the least \( k \) such that \( \frac{\chi(G)}{\chi(G)} \) is bounded by \( r(\ln r)^c \) for some \( c \)?

## 4 Diameter and Chromatic Number

In this section we study the relationship between \( \chi_r(G) \) and \( \chi(G) \) when \( G \) has small diameter. We first prove \( \chi_2(G) \leq \chi(G) + 2 \) when \( \text{diam}(G) = 2 \), regardless of whether \( G \) is regular, and we characterize when equality holds. The proof of the bound is easy, but the characterization of equality takes more work. Given a coloring \( f \) of a graph \( G \), say that a vertex is \( f \)-monochromatic if its neighbors have the same color under \( f \).

**Theorem 4.1.** If \( \text{diam}(G) = 2 \), then \( \chi_2(G) \leq \chi(G) + 2 \), with equality only when \( G \) is a complete bipartite graph or \( C_5 \).

**Proof.** The claim holds trivially for stars. For non-stars with minimum degree 1, the graph obtained by deleting all vertices of degree 1 has the same chromatic number, and a 2-dynamic coloring of it extends to a 2-dynamic coloring of the original graph. Hence we may assume \( \delta(G) \geq 2 \). We claim

\[ (*) \text{ If } f \text{ is a proper coloring of a graph with diameter 2, and } v \text{ is } f \text{-monochromatic with neighborhood of color } a, \text{ then } N(v) = \{ u : f(u) = a \}. \]

In particular, nonadjacent \( f \)-monochromatic vertices have the same neighborhood.

To prove \((*)\), note that the set of vertices with color \( a \) is independent, so a vertex outside \( N(v) \) with color \( a \) cannot have a common neighbor with \( v \). For the second statement, nonadjacent vertex must have a common neighbor, and then being \( f \)-monochromatic makes both adjacent to all vertices of that color.

We next prove the upper bound. Let \( f \) be a proper \( \chi(G) \)-coloring of a graph \( G \) with diameter 2; note that \( G \) is connected. If \( f \) is not 2-dynamic, then some vertex \( v \) is \( f \)-monochromatic, with color \( a \) on all of \( N(v) \). Modify \( f \) by giving a new color \( \alpha \) to \( v \) and another new color \( \beta \) to one vertex \( x \) in \( N(v) \).

The resulting coloring \( f' \) is a proper \( (\chi(G) + 2) \)-coloring. If \( f' \) is not 2-dynamic, then some vertex \( z \) is \( f' \)-monochromatic. By construction, \( z \) cannot be \( v \) or a neighbor of \( v \) (each neighbor of \( v \) has color \( \alpha \) on exactly one neighbor). If \( z \) is not a neighbor of \( v \), then by \((*)\) we have \( N(z) = N(v) \), and colors \( a \) and \( \beta \) both appear in \( N(z) \).

For the characterization of equality, note first that if no vertex of \( N(v) \) is \( f \)-monochromatic, then we do not need to introduce a new color on \( v \), and we obtain \( \chi_2(G) \leq \chi(G) + 1 \). Hence we may assume that two adjacent vertices \( v \) and \( u \) are \( f \)-monochromatic. If no vertex lies outside \( N(v) \cup N(u) \), then \( G \) is the complete bipartite graph with parts \( N(v) \) and \( N(u) \), since those sets are independent and \( \text{diam}(G) = 2 \).
Hence if $G$ is not a complete bipartite graph, then there is a vertex $w$ outside $N(v) \cup N(u)$. Let $a$ and $b$ be the colors on $N(v)$ and $N(u)$, respectively. Since $\text{diam}(G) = 2$, vertex $w$ has neighbors in both $N(v)$ and $N(u)$ and hence has a third color, $c$. Let $W$ be the set of all vertices with color $c$ under $f$. Let $U$ and $V$ be the set of all $f$-monochromatic vertices having the same color as $u$ and $v$, respectively; note that $U \subseteq N(v)$ and $V \subseteq N(u)$. By (*), there are no other $f$-monochromatic vertices.

All vertices in $U$ have the same neighborhood, as do all vertices in $V$. If $|U| > 1$, then we change the color of $u$ to $c$ and use a new color $d$ on $v$. This produces a 2-dynamic coloring with $\chi(G) + 1$ colors, since the neighbors of $u$ now have colors $a$ and $c$ in their neighborhood. Similarly, if $|V| > 1$ we can change the color of $v$ to $c$ and use $d$ on $u$.

Hence we may assume $|U| = |V| = 1$. The change still works unless $N(u)$ contains a vertex $x$ whose neighbors other than $u$ all have color $c$, and similarly $N(v)$ contains a vertex $y$ whose neighbors other than $v$ all have color $c$. In particular, $x$ and $y$ are not adjacent. Now changing $x$ and $y$ to a new color $d$ produces the desired 2-dynamic coloring of $G$ with $\chi(G) + 1$ colors, unless there is a vertex $w \in W$ whose only neighbors are $x$ and $y$.

In this case, reaching $N(u) - \{v, x\}$ in two steps from $w$ requires $N(u) - \{v, x\} \subseteq N(y)$, and similarly $N(v) - \{u, y\} \subseteq N(x)$. Since vertices of $N(x) - \{u\}$ and $N(y) - \{v\}$ all have color $c$, we conclude that $N(u) = \{v, x\}$ and $N(v) = \{u, y\}$. We are now left with $G = C_5$ unless $G$ has a vertex outside the 5-cycle induced by $w, x, u, v, y$ in order. Since $u$ and $v$ have no other neighbors, and all other neighbors of $x$ and $y$ have color $c$, reaching $u$ and $v$ in two steps requires that all the remaining vertices have neighborhood $\{x, y\}$. Now we use colors $1, 2, 3, 4$ in order on the path $(x, u, v, y)$ and use colors 2 and 3 on the remaining independent set, each at least once.

Theorem 4.2. If $G$ is a $k$-chromatic graph with diameter at most 3, then $\chi_2(G) \leq 3k$, and this bound is sharp when $k \geq 2$.

Proof. We begin with the sharpness construction. Let $H = K_{3k}$, and let $F$ be a subgraph of $H$ consisting of disjoint triangles $T_1, \ldots, T_k$. Form $G$ from $H$ by subdividing each edge of $F$. In $G$ the vertices belonging to $T_i$ can receive color $i$, and their neighbors of degree 2 can receive another color. In a 2-dynamic coloring, all the original vertices of $H$ must have distinct colors, so $3k$ colors are needed. The diameter is 3, because any two vertices of degree 2 have neighbors that are adjacent.

Now let $G$ be any $k$-chromatic graph with diameter at most 3. As observed in the introduction, $\chi_r(G) \leq kj$, where $j$ is the minimum number of colors in an $r$-strong coloring
of the hypergraph $H$ of vertex neighborhoods in $G$. More precisely, we give each vertex $v$ a color pair $(f(v), h(v))$, where $f$ is a proper $k$-coloring of $G$ and $h$ is an $r$-strong coloring of the subhypergraph $H'$ of $H$ whose edges are the edges of $H$ that do not already have $r$ colors under $f$. The first coordinate ensures that the resulting coloring $g$ of $G$ is proper, and the second coordinate ensures that it is $r$-dynamic.

Therefore, it suffices to show that when $\text{diam}(G) \leq 3$, the resulting hypergraph $H'$ always has a 2-strong coloring with three colors. Call a hypergraph with such a coloring good. Note first that every intersecting hypergraph is good, where a hypergraph is intersecting if any two of its edges intersect. Given an intersecting hypergraph, just use colors 1 and 2 on a minimal edge and use color 3 on all vertices not in that edge (in fact, we only need one minimal edge that intersects all other edges).

Given the proper $k$-coloring $f$ of $G$, we define $k$ subhypergraphs of $H'$. Let $V_i = \{v \in V(G): f(v) = i\}$. The subhypergraph $H_i$ has vertex set $V_i$, and its edges are the vertex neighborhoods in $G$ in which every vertex has color $i$. That is, $N(u) \in E(H_i)$ if $f(v) = i$ for all $v \in N(u)$. Since each vertex in $H_i$ has color $i$ under $f$, the vertex sets of these subhypergraphs are disjoint. Hence if each is good, then their union is good. We can define $h(v)$ arbitrarily for $v \notin V(H')$, because we do not need multiple colors in the second coordinate of $g$ on a neighborhood where the first coordinate already provides multiple colors.

It remains only to show that $H_i$ is an intersecting hypergraph when $\text{diam}(G) \leq 3$. Consider $x, y \in V(G)$ such that $N(x), N(y) \in E(H_i)$. If $xy \in E(G)$, then $y \in N(x)$ and $x \in N(y)$, and hence $f(x) = f(y) = i$, which contradicts that $f$ is a proper coloring of $G$. Hence $x$ and $y$ are not adjacent in $G$. Also, no edge of $G$ can join $N(x)$ and $N(y)$, since all vertices in $N(x) \cup N(y)$ have color $i$ under $f$. If $N(x) \cap N(y) = \emptyset$, then a shortest path in $G$ from $x$ to $y$ must pass through $N(x)$ and $N(y)$ and some other vertex between them. Hence the path has length at least 4, which contradicts $\text{diam}(G) \leq 3$. We conclude that $H_i$ is an intersecting hypergraph and hence is good, as desired.

These results are sharp in various ways. For larger $r$, there is no bound, not even on bipartite graphs with diameter 3 or on 3-chromatic graphs with diameter 2. Note that the only bipartite graphs with diameter 2 are complete bipartite graphs, where $r$-dynamic chromatic number does not exceed $2r$.

**Theorem 4.3.** For $3 \leq k < r$, the $r$-dynamic chromatic number is unbounded on the graphs with minimum degree $k + 1$ that are bipartite and have diameter 3, and also on those that are 3-colorable and have diameter 2.

**Proof.** Let $H$ be the incidence graph of the $k$-subsets of $[n]$, where $n > k + 1$. That is, $H$ is bipartite, with one part being $[n]$ and the other being the family of $k$-subsets, and element $j$ is adjacent to set $A$ if $j \in A$. Form $G$ by adding a single vertex $v$ adjacent to all the $k$-sets, giving them degree $k + 1$. The graph $G$ is bipartite, with the added vertex in the same part.
as [n]. Any two elements of [n] lie in a common k-set, so the distance between them is 2, and the distance between any k-set and an element not in it is 3. The added vertex has distance 2 from all of [n] and ensures distance 2 between any two k-sets. Hence \( \text{diam}(G) = 3 \).

In an \( r \)-dynamic coloring, the neighbors of any vertex with degree at most \( r \) receive distinct colors. In \( G \), any two vertices of \([n] \cup \{v\}\) have a common neighbor with degree at most \( r \), so \( \chi_r(G) \geq n + 1 \) (equality holds).

For a construction with diameter 2, let the added vertex \( v \) be adjacent also to all of \([n]\). Now \( v \) is a dominating vertex, so the diameter is 2, but the chromatic number increases to 3. However, each k-set vertex still has \( k + 1 \) neighbors, so the argument for \( \chi_r(G) \geq n + 1 \) remains the same. □

## 5 Cartesian Products

In this final section, we study \( r \)-dynamic coloring of cartesian products of graphs. We will study products of two paths or two cycles, but first we note a general upper bound on \( \chi_r(G \square H) \). The special case \( r = 2 \) was proved by Akbari, Ghanbari, and Jahanbekam [2], and the general result is proved by simply replacing 2 with \( r \) in their proof.

**Theorem 5.1.** Let \( G \) and \( H \) be graphs. If \( \delta(G) \geq r \), then \( \chi_r(G \square H) \leq \max\{\chi_r(G), \chi(H)\} \).

*Proof.* Let \( c \) be an \( r \)-dynamic coloring of \( G \) using colors \( \{1, \ldots, \chi_r(G)\} \), and let \( c' \) be a proper \( \chi(H) \)-coloring of \( H \) using integers. Let \( m = \max\{\chi_r(G), \chi(H)\} \). Define a coloring \( f \) on \( G \square H \) by \( f(u, v) = c(u) + c'(v) \mod m \). Since \( c \) and \( c' \) are proper colorings, \( f \) is a proper coloring. Also \( f \) is \( r \)-dynamic, since \( |f(N((u, v)))| \geq |c(N_G(u))| \geq \min\{r, d_G(u)\} = r \). □

For cartesian products involving paths and cycles, some exact results are available. Akbari, Ghanbari, and Jahanbekam [2] showed that \( \chi_2(P_m \square P_n) = 4 \) and that \( \chi_2(C_m \square C_n) = 3 \) when \( 3 \mid mn \), and that otherwise \( \chi_2(C_m \square C_n) = 4 \). They also determined \( \chi_2(C_m \square P_n) \).

Since these graphs have maximum degree 4, by Observation 1.1 the \( r \)-dynamic chromatic number equals the 4-dynamic chromatic number when \( r \geq 4 \). Hence we confine our attention to \( r \in \{3, 4\} \). The minimum degree of the factors is too small to apply Theorem 5.1, so upper bounds require constructions. We determine \( \chi_r(P_m \square P_n) \) for all \((m, n)\) except when \( r = 3 \) and \( mn \equiv 2 \mod 4 \).

**Theorem 5.2.** If \( m \) and \( n \) are at least 2, then

\[
\chi_4(P_m \square P_n) = \begin{cases} 4 & \text{if } \min\{m, n\} = 2 \\ 5 & \text{otherwise} \end{cases} \quad \text{and} \quad \chi_3(P_m \square P_n) = \begin{cases} 4 & \min\{m, n\} = 2 \\ 4 & m \text{ and } n \text{ are both even} \end{cases}
\]

*Proof.* In each case, the lower bound arises from vertex neighborhoods as in Observation 1.2: \( \chi_r(G) \geq \min\{\Delta(G), r\} + 1 \). For the upper bounds, express \( V(P_m \square P_n) \) as \( \{(i, j): 0 \leq i \leq m - 1, 0 \leq j \leq n - 1\} \), with vertices adjacent when they differ by 1 in one coordinate.
First consider $\chi_4(P_m \square P_n)$ with $\min\{m, n\} > 2$. Define a coloring $c$ on $V(P_m \square P_n)$ by $c(i, j) = i + 2j \mod 5$. By construction, $c$ is a proper 5-coloring of $P_m \square P_n$. It is also 4-dynamic, because the neighbors of any vertex have distinct colors. Thus $\chi_4(P_m \square P_n) = 5$.

For $\min\{m, n\} = 2$ and $r \geq 3$, we have $\Delta(P_m \square P_n) = 3$, so $\chi_r(P_m \square P_n) \geq 4$; the coloring shown below for $m = 2$ achieves equality and illustrates the coloring $h$ defined below.

\[
\begin{array}{ccccc}
0 & 1 & 2 & 3 & 0 \\
2 & 3 & 0 & 1 & 2 \\
\end{array}
\]

Now consider $\chi_3(P_m \square P_n)$ when both $m$ and $n$ are even; at least four colors are needed. Define $g$ by $g(4k) = 0$, $g(4k + 1) = 2$, $g(4k + 2) = 1$, and $g(4k + 3) = 3$ for $k \in \mathbb{Z}$. Define a coloring $h$ on $P_m \square P_n$ by setting $h(i, j) = g(i) + j \mod 4$ when $i$ or $i - 1$ is divisible by 4 and $h(i, j) = g(i) - j \mod 4$ otherwise. By construction, $h$ is a proper 4-coloring of $P_m \square P_n$. Also $h$ is 3-dynamic; at least three colors appear in each neighborhood.

Theorem 5.3. Always $\chi_3(P_m \square P_n) \leq 5$, with equality when one of $m$ or $n$ is odd and the other is not congruent to 2 modulo 4. In any 3-dynamic 4-coloring of $P_m \square P_n$, let $a_{i,j}$ be the color of the vertex $(i, j)$ in row $i$ and column $j$, numbered from the top left. The first two rows and columns and the last two rows and columns are uniquely determined by the colors $a_{1,1}$, $a_{1,2}$, and $a_{2,1}$.

Proof. The upper bound follows from the construction in Theorem 5.2 that $\chi_4(P_m \square P_n) \leq 5$, since always $\chi_r(G) \leq \chi_{r+1}(G)$.

For the claim of equality, we first prove the second statement. Let $A$ be the matrix with color $a_{i,j}$ in position $(i, j)$, representing the coloring. We may assume by symmetry that $m$ (the number of rows in $A$) is odd. The four vertices in the upper left have distinct colors, since the vertices on the edges of the matrix have degree at most 3. We may let $a_{1,1} = a$, $a_{1,2} = b$, $a_{2,1} = c$, and $a_{2,2} = d$. The neighbors of a vertex on the edge must have the other three colors. Repeatedly using this observation determines the first two rows and first two columns. Once the argument for the first two rows and columns reaches their ends, the same argument determines the elements of the last two rows and columns.

The diagram below, in the two cases $m \equiv 1 \mod 4$ and $m \equiv 3 \mod 4$, incorporates all the cases for $m$. Note that in the bottom row the first two elements agree with the top row when $m \equiv 1 \mod 4$ and reverse those two elements when $m \equiv 3 \mod 4$. The diagram
shows that the last two columns exhibit the same behavior.

\[
\begin{array}{cccccccccccccccc}
\text{a} & \text{b} & \text{c} & \text{d} & \text{a} & \text{b} & \text{c} & \text{d} & \text{a} & \text{b} & \text{c} & \text{d} & \text{a} & \text{b} & \text{c} & \text{d} \\
\text{a} & \text{b} & \text{c} & \text{d} & \text{a} & \text{b} & \text{c} & \text{d} & \text{a} & \text{b} & \text{c} & \text{d} & \text{a} & \text{b} & \text{c} & \text{d} \\
\text{b} & \text{a} & \text{b} & \text{a} & \text{b} & \text{a} & \text{b} & \text{a} & \text{b} & \text{a} & \text{b} & \text{a} & \text{b} & \text{a} & \text{b} & \text{a} \\
\text{c} & \text{d} & \text{a} & \text{b} & \text{c} & \text{d} & \text{a} & \text{b} & \text{c} & \text{d} & \text{a} & \text{b} & \text{c} & \text{d} & \text{a} & \text{b} \\
\text{1} & 1 & 4 & 4 & 3 & 3 & 2 & 2 & 1 & 1 & 4 & 4 & 3 & 3 & 2 & 2
\end{array}
\]

This property of the last two columns occurs for each congruence class of \( n \) modulo 4. The numbers below the grid designate where the rows end when \( n \) is congruent to 1, 4, 3, or 2, respectively. In the first three cases, the relationship between the top row and bottom row is not as would be required by the last two columns if the rows ended there. Hence in those cases no 3-dynamic 4-coloring can exist.

The diagrams in Theorem 5.3 show that consistency of the colors on the borders and their neighbors can be achieved when \( m \) is odd and \( n \equiv 2 \mod 4 \), leaving open the possibility of completing a 3-dynamic 4-coloring in such cases. Using further structural arguments, we were able to forbid this when \( m \) or \( n \) is at most 15. Subsequently, Kang, Muller and West [8] found a proof for all \((m, n)\) with \( mn \equiv 2 \mod 4 \) that \( \chi_3(P_m \square P_n) > 4 \). Their result completes the following theorem.

**Theorem 5.4.** When \( m \) and \( n \) are not both even, \( \chi_3(P_m \square P_n) = 5 \).

Finally, consider \( \chi_r(C_m \square C_n) \) for \( r \geq 3 \). Since \( \chi_r(C_m \square C_n) = \chi_r(C_n \square C_m) \), we may assume that the remainder upon dividing \( m \) by 4 is no larger than when dividing \( n \) by 4.

**Theorem 5.5.** Always \( \chi_3(C_m \square C_n) \geq 4 \). For \( m \equiv s \mod 4 \) and \( n \equiv t \mod 4 \) with \( 0 \leq s \leq t \leq 3 \), equality holds when \( s = 0 \) and \( t \neq 3 \).

**Proof.** Since \( C_m \square C_n \) is 4-regular, \( \chi_3(C_m \square C_n) \geq 4 \) by Observation 1.1. For the upper bound, we construct colorings. Write \( V(C_m \square C_n) \) as \( \mathbb{Z}_m \times \mathbb{Z}_n \), with vertices adjacent when they agree in one coordinate and differ by 1 in the other.

Recall \( g \) and \( h \) from Theorem 5.2: \( g(4k) = 0, g(4k + 1) = 2, g(4k + 2) = 1 \), and \( g(4k + 3) = 3 \) for \( k \in \mathbb{Z} \). Also \( h(i, j) = g(i) + j \mod 4 \) when \( i \) or \( i - 1 \) is divisible by 4, and \( h(i, j) = g(i) - j \mod 4 \) otherwise. Indexing of the rows and columns begins with 0. Below we illustrate \( h \) and two modifications of \( h \) to be used when \( s = 0 \). Also let \( A \) denote the 4-by-4 matrix appearing in the first four columns of the first matrix below. Note that \( h \) is a tiling by copies of \( A \) when \( s = t = 0 \), and otherwise it uses portions of \( A \) in the rows and

\[
\begin{array}{cccccccccccccccc}
\text{a} & \text{b} & \text{c} & \text{d} & \text{a} & \text{b} & \text{c} & \text{d} & \text{a} & \text{b} & \text{c} & \text{d} & \text{a} & \text{b} & \text{c} & \text{d} \\
\text{a} & \text{b} & \text{c} & \text{d} & \text{a} & \text{b} & \text{c} & \text{d} & \text{a} & \text{b} & \text{c} & \text{d} & \text{a} & \text{b} & \text{c} & \text{d} \\
\text{b} & \text{a} & \text{b} & \text{a} & \text{b} & \text{a} & \text{b} & \text{a} & \text{b} & \text{a} & \text{b} & \text{a} & \text{b} & \text{a} & \text{b} & \text{a} \\
\text{d} & \text{c} & \text{a} & \text{b} & \text{d} & \text{c} & \text{a} & \text{b} & \text{d} & \text{c} & \text{a} & \text{b} & \text{d} & \text{c} & \text{a} & \text{b} \\
\text{a} & \text{b} & \text{a} & \text{b} & \text{a} & \text{b} & \text{a} & \text{b} & \text{a} & \text{b} & \text{a} & \text{b} & \text{a} & \text{b} & \text{a} & \text{b} \\
\text{c} & \text{d} & \text{b} & \text{a} & \text{c} & \text{d} & \text{b} & \text{a} & \text{c} & \text{d} & \text{b} & \text{a} & \text{c} & \text{d} & \text{b} & \text{a} \\
\text{b} & \text{a} & \text{c} & \text{d} & \text{b} & \text{a} & \text{c} & \text{d} & \text{b} & \text{a} & \text{c} & \text{d} & \text{b} & \text{a} & \text{c} & \text{d} \\
\text{1} & 1 & 4 & 4 & 3 & 3 & 2 & 2 & 1 & 1 & 4 & 4 & 3 & 3 & 2 & 2
\end{array}
\]
columns after the last multiple of 4.

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 0 & 1 \\
2 & 3 & 0 & 1 & 2 & 3 \\
1 & 0 & 3 & 2 & 1 & 0 \\
3 & 2 & 1 & 0 & 3 & 2 \\
\end{array}
\quad
\begin{array}{cccccc}
0 & 1 & 2 & 3 & | & 1 \\
2 & 3 & 0 & 1 & | & 3 \\
1 & 0 & 3 & 2 & | & 0 \\
3 & 2 & 1 & 0 & | & 2 \\
\end{array}
\quad
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 0 & 1 \\
2 & 3 & 0 & 1 & 2 & 3 \\
1 & 0 & 3 & 2 & 1 & 0 \\
3 & 2 & 1 & 0 & 3 & 2 \\
\end{array}
\]

As shown above on the left, each column is periodic, so each vertex has vertically neighboring colors distinct and different from its own. When \( t = 0 \), the coloring is 4-dynamic (for the same reason) and hence also 3-dynamic. When \( t = 2 \), the coloring is still proper, but the vertices in the last column have horizontal neighbors with the same color; the coloring is still 3-dynamic.

When \( t = 1 \), modify \( h \) by changing the colors on column \( n - 1 \) (the last column) to agree with those on column 1 (the second column). Colors on vertices in the last column now differ by 2 from the color to their left, so the coloring is proper. Vertices in the last two columns have three distinct colors in their neighborhoods; other vertices have four.

In the remaining cases, explicit constructions yield \( \chi_3(C_m \boxtimes C_n) \leq 6 \) (see [15]), but we have not determined the optimal values. Similarly, for \( r \geq 4 \) we have \( \chi_r(C_m \boxtimes C_n) = \chi_4(C_m \boxtimes C_n) \geq 5 \), with equality when \( m \) and \( n \) are both divisible by 5, and explicit constructions yield \( \chi_4(C_m \boxtimes C_n) \leq 9 \) (see [15]). Note that \( \chi_4(C_3 \boxtimes C_3) = 9 \), since \( (C_3 \boxtimes C_3)^2 \) is a complete graph.

**References**


