

INTRODUCTION TO RAMANUJAN'S LOST NOTEBOOK

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In the spring of 1976, the first author visited Trinity College Library at Cambridge University. Dr. Lucy Slater had suggested to him that there were materials deposited there from the estate of the late G.N. Watson that might be of interest to him. In one box of materials from Watson's estate, Andrews found several items written by Srinivasa Ramanujan. The most interesting item in this box was a manuscript written on 138 sides in Ramanujan's distinctive handwriting. The sheets contained over six hundred mathematical formulas without proofs. Although technically not a notebook, and although technically not "lost," as we shall see in the sequel, it was natural in view of the fame of Ramanujan's (earlier) notebooks [5] to call this manuscript *Ramanujan's lost notebook*. Almost surely, this manuscript, or at least most of it, was written during the last year of Ramanujan's life, after his return to India from England. We do not possess a bona fide proof of this claim, but we shall later present considerable evidence for it.

The manuscript contains no introduction or covering letter. In fact, there are hardly any words in the manuscript. There are a few marks evidently made by a cataloguer, and there are also a few remarks in the handwriting of G.H. Hardy. Undoubtedly, the most famous objects examined in the lost notebook are the *mock theta functions*, about which more will be said later. Concerning this manuscript, Ms. Rosemary Graham, manuscript cataloguer of the Trinity College Library, remarked, "... the notebook and other material was discovered among Watson's papers by Dr. J.M. Whittaker, who wrote the obituary of Professor Watson for the Royal Society. He passed the papers to Professor R.A. Rankin of Glasgow University, who, in December 1968, offered them to Trinity College so that they might join the other Ramanujan manuscripts already given to us by Professor Rankin on behalf of Professor Watson's widow." Since her late husband had been a fellow and scholar at Trinity College and had had an abiding, lifelong affection for Trinity College, Mrs. Watson agreed with Rankin's suggestion that the library at Trinity College would be the best place to preserve her husband's papers. Since Ramanujan had also been a fellow at Trinity College, Rankin's suggestion was even more appropriate.

The natural, burning question now is, *How did this manuscript of Ramanujan come into Watson's possession?* We think that the manuscript's history can be traced.

HISTORY OF THE LOST NOTEBOOK

After Ramanujan died on April 26, 1920, his notebooks and unpublished papers were given by his widow, Janaki, to the University of Madras. (It should be remarked that in a conversation with the second author, Janaki had told him that during the funeral of her late husband, many of his papers were stolen by two certain people. These

stolen papers have never been located.) Also at that time, Hardy strongly advocated bringing together all of Ramanujan's manuscripts, both published and unpublished, for publication. On August 30, 1923, Francis Dewsbury, the registrar at the University of Madras, wrote to Hardy informing him that [1, p. 266]:

I have the honour to advise despatch to-day to your address per registered and insured parcel post of the four manuscript note-books referred to in my letter No. 6796 of the 2nd idem.

I also forward a packet of miscellaneous papers which have not been copied. It is left to you to decide whether any or all of them should find a place in the proposed memorial volume. Kindly preserve them for ultimate return to this office.

(The notebooks were returned to Madras, but Hardy evidently kept all the miscellaneous papers.) Although no accurate record of this material exists, the amount sent to Hardy was doubtless substantial. It is therefore highly likely that this "packet of miscellaneous papers" contained the aforementioned "lost notebook." Rankin, in fact, opines [2, p. 124]:

It is clear that the long MS represents work of Ramanujan subsequent to January 1920 and there can therefore be little doubt that it constitutes the whole or part of the miscellaneous papers dispatched to Hardy from Madras on 30 August 1923.

Further details can be found in Rankin's accounts of Ramanujan's unpublished manuscripts [8], [1, pp. 120–123], [2, pp. 117–142].

In 1934, Hardy passed on to Watson a considerable amount of his material on Ramanujan. However, it appears that either Watson did not possess the "lost" notebook in 1936 and 1937 when he published his papers [10], [11] on mock theta functions, or he had not examined it thoroughly. In any event, Watson [10, p. 61], [1, p. 330] writes that he believes that Ramanujan was unaware of certain third order mock theta functions and their transformation formulas. But, in his lost notebook, Ramanujan did indeed examine these functions and their transformation formulas. Watson's interest in Ramanujan's mathematics waned in the late 1930s, and Hardy died in 1947. In conclusion, sometime between 1934 and 1947 and probably closer to 1947, Hardy gave Watson the manuscript we now call the "lost notebook." More will be said in the sequel about further contents of the lost notebook.

Watson devoted about 10 to 15 years of his research to Ramanujan's work, with over 30 papers having their genesis in Ramanujan's mathematics, in particular, his notebooks and the letters he wrote to Hardy from India. Watson was Mason Professor of Pure Mathematics at the University of Birmingham for most of his career, retiring in 1951. He died in 1965 at the age of 79. Rankin who succeeded Watson as Mason Professor of Pure Mathematics in Birmingham but who had since become Professor of Mathematics at the University of Glasgow, was asked to write an obituary of Watson for the London Mathematical Society. Rankin writes [8], [2, p. 120]:

For this purpose I visited Mrs Watson on 12 July 1965 and was shown into a fair-sized room devoid of furniture and almost knee-deep in manuscripts covering the floor area. In the space of one day I had time only to make

a somewhat cursory examination, but discovered a number of interesting items. Apart from Watson's projected and incomplete revision of Whittaker and Watson's *Modern Analysis* in five or more volumes, and his monograph on *Three decades of midland railway locomotives*, there was a great deal of material relating to Ramanujan, including copies of Notebooks 1 and 2, his work with B.M. Wilson on the Notebooks and much other material. . . . In November 19 1965 Dr J.M. Whittaker who had been asked by the Royal Society to prepare an obituary notice [12], paid a similar visit and unearthed a second batch of Ramanujan material. A further batch was given to me in April 1969 by Mrs Watson and her son George.

A more colorful rendition of Whittaker's visit with Mrs. Watson was described in a letter of August 15, 1979, to Andrews [1, p. 304]:

When the Royal Society asked me to write G.N. Watson's obituary memoir I wrote to his widow to ask if I could examine his papers. She kindly invited me to lunch and afterwards her son took me upstairs to see them. They covered the floor of a fair sized room to a depth of about a foot, all jumbled together, and were to be incinerated in a few days. One could only make lucky dips and, as Watson never threw away anything, the result might be a sheet of mathematics but more probably a receipted bill or a draft of his income tax return for 1923. By an extraordinary stroke of luck one of my dips brought up the Ramanujan material which Hardy must have passed on to him when he proposed to edit the earlier notebooks.

(That Watson's papers "were to be incinerated in a few days" seems fanciful.) Rankin dispatched Watson's and Ramanujan's papers to Trinity College in three batches on November 2, 1965; December 26, 1968; and December 30, 1969, with the Ramanujan papers being in the second shipment. Rankin did not realize the importance of Ramanujan's papers, and so when he wrote Watson's obituary [7] for the *Journal of the London Mathematical Society*, he did not mention any of Ramanujan's manuscripts. Thus, for almost eight years, Ramanujan's "lost notebook" and some fragments of papers by Ramanujan lay in the library at Trinity College, known only to a few of the library's cataloguers, Rankin, Mrs. Watson, Whittaker, and perhaps a few others. The 138-page manuscript waited there until Andrews found it and brought it before the mathematical public in the spring of 1976. It was not until the centenary of Ramanujan's birth on December 22, 1987, that Narosa Publishing House in New Delhi published in photocopy form Ramanujan's lost notebook and his other unpublished papers [6].

THE ORIGIN OF THE LOST NOTEBOOK

Having detailed the probable history of Ramanujan's lost notebook, we return now to our earlier claim that the lost notebook emanates from the last year of Ramanujan's life. On February 17, 1919, Ramanujan returned to India after almost five years in England, the last two being confined to nursing homes. Despite the weakening effects of

his debilitating illness, Ramanujan continued to work on mathematics. Of this intense mathematical activity, up to the discovery of the lost notebook, the mathematical community knew only of the mock theta functions. These functions were described in Ramanujan's last letter to Hardy, dated January 12, 1920 [4, pp. xxix–xxx, 354–355], [1, pp. 220–223], where he wrote:

I am extremely sorry for not writing you a single letter up to now
I discovered very interesting functions recently which I call “Mock” ϑ -functions. Unlike the “False” ϑ -functions (studied partially by Prof. Rogers in his interesting paper) they enter into mathematics as beautifully as the ordinary theta functions. I am sending you with this letter some examples.

In this letter, Ramanujan defines four third order mock theta functions, ten fifth order functions, and three seventh order functions. He also includes three identities satisfied by the third order functions and five identities satisfied by his first five fifth order functions. He states that the other five fifth order functions also satisfy similar identities. In addition to the definitions and formulas stated by Ramanujan in his last letter to Hardy, the lost notebook contains further discoveries of Ramanujan about mock theta functions. In particular, it contains the five identities for the second family of fifth order functions that were only mentioned but not stated in the letter.

We hope that we have made the case for our assertion that the lost notebook was composed during the last year of Ramanujan's life, when, by his own words, he discovered the mock theta functions. In fact, only a fraction (perhaps 5%) of the notebook is devoted to the mock theta functions themselves.

THE CONTENT OF THE LOST NOTEBOOK

The next fundamental question is, *What is in Ramanujan's lost notebook besides mock theta functions?* A majority of the results fall under the purview of q -series. These include mock theta functions, theta functions, partial theta function expansions, false theta functions, identities connected with the Rogers–Fine identity, several results in the theory of partitions, Eisenstein series, modular equations, the Rogers–Ramanujan continued fraction, other q -continued fractions, asymptotic expansions of q -series and q -continued fractions, integrals of theta functions, integrals of q -products, and incomplete elliptic integrals. Other continued fractions, other integrals, infinite series identities, Dirichlet series, approximations, arithmetic functions, numerical calculations, Diophantine equations, and elementary mathematics are some of the further topics examined by Ramanujan in his lost notebook.

The Narosa edition [6] contains further unpublished manuscripts, fragments of both published and unpublished papers, letters to Hardy written from nursing homes, and scattered sheets and fragments. The three most famous of these unpublished manuscripts are those on the partition function and Ramanujan's tau function, forty identities for the Rogers–Ramanujan functions, and the unpublished remainder of Ramanujan's published paper on highly composite numbers [3], [4, pp. 78–128].

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