PROBABILITY GROUPS

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The down-to-earth predecessor of this finitely additive i.p.m. is the density function, which, for \( \mathbb{Z} \), can be defined by

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d(A) := \lim_{n \to \infty} \frac{1}{|[-n, n]|} |A \cap [-n, n]|
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for \( A \subseteq \mathbb{Z} \), whenever the limit exists.
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d(A) := \lim_{n \to \infty} \frac{1}{|[-n, n]|} |A \cap [-n, n]| = \lim_{n \to \infty} \frac{1}{|[-n, n]|} \sum_{k=-n}^{n} \mathbb{1}_{A}(k),
\]

for \( A \subseteq \mathbb{Z} \), whenever the limit exists.
This averaging gadget is used to express, for example, the mean ergodic theorem: for an ergodic action $\mathbb{Z} \curvearrowright (X, \nu)$ via $T : X \rightarrow X$ and any $f \in L^2(X, \nu)$,

$$\lim_{n \rightarrow \infty} \frac{1}{|[-n, n]|} \sum_{k=-n}^{n} T^k(f) = \int_X f(x) d\nu(x)$$

**time average = space average**
Integrating over the group — wishful thinking

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\lim_{n \to \infty} \frac{1}{|[−n, n]|} \sum_{k=−n}^{n} T^k(f) = \int_X f(x) d\nu(x)
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- Now, wouldn’t it be good if $\lim_{n \to \infty} \frac{1}{|[−n, n]|} \sum_{k=−n}^{n}$ could be replaced by a genuine $\int_G$?
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Now, wouldn’t it be good if \( \lim_{n \to \infty} \frac{1}{|[-n, n]|} \sum_{k=-n}^{n} \) could be replaced by a genuine \( \int_G \)?

In a certain sense, this can actually be done using ultraproducts!
Let $\alpha$ be an ultrafilter on $\mathbb{N}$, i.e. a $\{0, 1\}$-valued finitely additive probability measure defined on all subsets of $\mathbb{N}$.
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Ultraproducts overview

- Let $\alpha$ be an ultrafilter on $\mathbb{N}$, i.e. a $\{0, 1\}$-valued finitely additive probability measure defined on all subsets of $\mathbb{N}$. Further assume that $\alpha$ is non-principal, i.e. not a Dirac point measure.

- Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of sets, and for any $x, y \in \prod_{n \in \mathbb{N}} X_n$, put 
  $x =_\alpha y :\iff x(n) = y(n)$ for $\alpha$-a.e. $n \in \mathbb{N}$.
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The space $\prod_{n \in \mathbb{N}} X_n / =_\alpha$ is called the ultraproduct of $(X_n)_{n \in \mathbb{N}}$ over $\alpha$. 

Theorem (Łoś) A first-order statement is true in the ultraproduct $X$ if and only if it is true in $X_n$ for $\alpha$-a.e. $n \in \mathbb{N}$. 

If each $X_n$ is a group $(G_n, e_n, \cdot_n)$, the ultraproduct is also a group with identity $\lbrack (e_n)_{n \in \mathbb{N}} \rbrack_\alpha$ and coordinate-wise multiplication.
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More generally, we have the so-called transfer principle:

**Theorem (Łoś)**

A first-order statement is true in the ultraproduct $X$ if and only if it is true in $X_n$ for $\alpha$-a.e. $n \in \mathbb{N}$.
Furthermore, if each $X_n$ is a probability space $(X_n, B_n, \mu_n)$, then the ultraproduct is too with

$$\lim_{n \to \alpha} \mu_n(B_n).$$
Finitely additive measures $\xrightarrow{\text{ultraproduct}}$ countably additive measure

Furthermore, if each $X_n$ is a probability space $(X_n, \mathcal{B}_n, \mu_n)$, then the ultraproduct is too with

- the $\sigma$-algebra $\mathcal{B}$ generated by the box-sets: $\left[ \prod_{n \in \mathbb{N}} B_n \right]_\alpha$, for $B_n \in \mathcal{B}_n$, and the so-called Loeb measure $\lambda$ on it defined by $\lambda(\cdots) = \lim_{n \to \alpha} \mu_n(B_n)$.

The magic of ultraproducts is that even if the $\mu_n$'s were only finitely additive, the Loeb measure $\lambda$ would still be countably additive!

Thus, we could start with an amenable group $\Gamma$ equipped with a finitely additive i.p.m. $\mu$, take its ultrapower $G := \Gamma^\mathbb{N} / = \alpha$ and get a much larger group $G$ with a genuine countably additive i.p.m. $\lambda$.

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- Thus, we could start with an amenable group $\Gamma$ equipped with a finitely additive i.p.m. $\mu$, take its ultrapower $G := \Gamma^\mathbb{N} / \equiv_\alpha$ and get a much larger group $G$ with a genuine countably additive i.p.m. $\lambda$.

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Moreover, for any $A \subseteq \Gamma$, $\mu(A) = \lambda\left(\left[ A^\mathbb{N} \right]_{\alpha}\right)$. 
The mean ergodic theorem for this $G$ now looks like this:

**Mean Ergodic Theorem**

Let $a : G \curvearrowright (X, \nu)$ be a measure-preserving ergodic action of $G$ on a probability space $(X, \nu)$. Then, for every $f \in L^2(X, \nu)$ and $\nu$-a.e. $x \in X$,

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**Proof.**
Application 1: trivializing the mean ergodic theorem

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This immediately implies the usual statement of the mean ergodic theorem for $\Gamma$ (with the awkward averaging gadget).
Application 2

Alternative to Furstenberg correspondence principle
Recurrence in countable groups

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*Any subset $A \subseteq \mathbb{Z}$ of positive upper density contains arbitrarily long arithmetic progressions.* In other words, $\forall k \geq 1 \exists n \in \mathbb{N}$

$$A \cap (A - n) \cap (A - 2n) \cap ... \cap (A - kn) \neq \emptyset.$$
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[Table: Theorem (Szemerédi)]
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- Think of $\overline{d}$ as a finitely subadditive invariant probability measure on $\Gamma$.
- The object of study here is the measure-preserving action $\Gamma \curvearrowright (\Gamma, \overline{d})$ by right translation.
- However, this dynamical system is hard to work with because of finite additivity.
Furstenberg correspondence

Luckily, Furstenberg came up with a way of translating this funny dynamical system $\Gamma \acts (\Gamma, \overline{d})$ to a standard one $\Gamma \acts^\alpha (X, \nu)$, where
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- there is a standard measure-preserving system $\Gamma \curvearrowright \alpha (X_A, \nu_A)$ with $A' \subseteq X_A$ such that for any $\gamma_1, \gamma_2, \ldots, \gamma_k \in \Gamma$,

$$
\overline{d}(A \cap A\gamma_1^{-1} \cap \ldots \cap A\gamma_k^{-1}) \geq \nu_A(A' \cap (\gamma_1^{-1} \cdot \alpha A') \cap \ldots \cap (\gamma_k^{-1} \cdot \alpha A')).
$$
Multiple Recurrence

**Theorem (Furstenberg–Katznelson)**

*If* $\Gamma$ *is abelian, then for any (countably additive) dynamical system* $\Gamma \curvearrowright \alpha (X, \nu)$, *any* $B \subseteq X$ *with* $\nu(B) > 0$ *and any* $\gamma_1, \gamma_2, \ldots, \gamma_k \in \Gamma$, *there is* $n \in \mathbb{N}$,  

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$$\nu (B \cap (\gamma_1^{-n} \cdot \alpha B) \cap \ldots \cap (\gamma_k^{-n} \cdot \alpha B)) > 0.$$ 

Thus, for $\Gamma = \mathbb{Z} = \langle T \rangle$, *we take* $\gamma_i = T^i$ *and get*

$$\nu (B \cap T^{-n}(B) \cap \ldots \cap T^{-kn}(B)) > 0,$$

*which implies Szemerédi’s theorem.*
Alternative approach: changing the group

- In the Furstenberg correspondence, only the action space changes, from $\Gamma \acts (\Gamma, \bar{d})$ to $\Gamma \acts (X_A, \nu_A)$, while the acting group $\Gamma$ stays the same.
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Thus, we obtain the following correspondence principle:

- For any $A \subseteq \Gamma$ of positive upper density, taking the ultrapower $A' := A^\mathbb{N} / \! \! =\alpha \subseteq G$, we have
  $$\overline{d}(A) = \lambda(A');$$
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- and the right translation action $\Gamma \ltimes (\Gamma, \overline{d})$ is naturally replaced by the right translation action $G \ltimes (G, \lambda)$.

Thus, we obtain the following correspondence principle:

- For any $A \subseteq \Gamma$ of positive upper density, taking the ultrapower $A' := A^\mathbb{N} / \alpha \subseteq G$, we have
  \[ \overline{d}(A) = \lambda(A'); \]
- moreover, viewing $\Gamma$ inside $G$, we have that for any $\gamma_1, \gamma_2, \ldots, \gamma_k \in \Gamma$,
  \[ \overline{d}(A \cap A\gamma_1^{-1} \cap \ldots \cap A\gamma_k^{-1}) \geq \lambda(A' \cap A'\gamma_1^{-1} \cap \ldots \cap A'\gamma_k^{-1}). \]
The class of probability groups
An annoying complication

Our goal is to define a class of groups equipped with an invariant (countably additive) probability measure so that it includes all compact groups equipped with the Haar measure; is closed under taking ultraproducts. However, ultraproducts cause complications with the measurability of the group operation...

Even for finite groups $G_n$ with their normalized counting measures $\mu_n$, the multiplication is trivially measurable wrt the $\sigma$-algebra $B_n \times B_n$, where $B_n = \mathcal{P}(G_n)$.

Thus, if $(G, B, \mu) = \prod_{n \to \alpha} (G_n, B_n, \mu_n)$, then the multiplication operation in $G$ is measurable wrt $B_2 = \prod_{n \to \alpha} B_n \times B_n$, but it may not be measurable with respect to $B \times B$. 


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Probability groups

- Let $G$ be a group (possibly very uncountable),
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We call the tuple $(G, (\mathcal{B}(n))_{n \geq 1}, (\mu(n))_{n \geq 1})$ a probability group if

1. The multiplication operation $\cdot : (G^2, \mathcal{B}(2)) \to (G, \mathcal{B})$ is measurable; in fact, all the word-multiplication maps $(G^n, \mathcal{B}(n)) \to (G^k, \mathcal{B}(k))$ are measurable.

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Examples of probability groups

- Compact Hausdorff groups are probability groups.
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▶ Compact Hausdorff groups are probability groups.
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Double recurrence in quasirandom groups
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Examples

- The left and right translation actions $G \curvearrowright \ell^2(G, \mu)$ and $G \curvearrowright r^2(G, \mu)$.
- The conjugation action $G \curvearrowright c^2(G, \mu)$.

▶ Any measure-preserving action $a : G \curvearrowright (X, \nu)$ lifts to a unitary action $G \curvearrowright L^2(X, \nu)$ by $g \cdot a f(x) := f(g^{-1} \cdot a x)$.

▶ For $f \in L^2(X, \nu)$, let $P_a(f)$ denote the orthogonal projection of $f$ onto the subspace $L^2_{a}(X, \nu)$ of invariant functions.
Measure-preserving actions of probability groups

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We now recall the notion of weak mixing for \( \mathbb{Z} \), but it is similarly defined for any countable amenable group.
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A measure-preserving ergodic action \( \mathbb{Z} \curvearrowright (X, \nu) \) via \( T : X \to X \) is called weakly mixing if for any \( A, B \subseteq X \), we have

\[
\lim_{n \to \infty} \frac{1}{|[-n, n]|} \sum_{k=-n}^{n} |\nu(A \cap T^{-k}(B)) - \nu(A)\nu(B)| = 0.
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Overview of mixing actions of countable amenable groups

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- More generally, without the ergodicity assumption, a measure-preserving action $\mathbb{Z} \curvearrowright (X, \nu)$ is called **weakly mixing** if for any $f_0, f_1 \in L^2(X, \nu)$:

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A measure-preserving action \( a : G \curvearrowright X \) of a probability group \( (G, \mu) \) on a probability space \( (X, \nu) \) is called **mixing along** \( \mu \) (or just **mixing**) if for any \( f_1, f_2 \in L^2(X, \nu) \), we have:

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**Definition (Gowers)**

For fixed $D \in \mathbb{N}$, a finite group $G$ is called $D$-quasirandom if it doesn’t have any nontrivial unitary representations of dimension less than $D$.

**Examples**

(Gowers) The alternating group $A_n$ is $(n - 1)$-quasirandom.

(Gowers) More generally, if $G$ is perfect (i.e. $[G, G] = G$) and has no normal subgroups of index less than $m$, then $G$ is $\sqrt{\log m}/2$-quasirandom.

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**Definition (Gowers)**
For fixed $D \in \mathbb{N}$, a finite group $G$ is called $D$-quasirandom if it doesn’t have any nontrivial unitary representations of dimension less than $D$.

**Examples**
- (Gowers) The alternating group $A_n$ is $(n - 1)$-quasirandom.
- (Gowers) More generally, if $G$ is perfect (i.e. $[G, G] = G$) and has no normal subgroups of index less than $m$, then $G$ is $\sqrt{\log m}/2$-quasirandom.
- (Frobenius) $SL_2(F_p)$ is $\frac{p-1}{2}$-quasirandom.
A measure-preserving action \( G \ltimes (X, \nu) \) of a probability group \((G, \mu)\) (not necessarily finite) is called \( \varepsilon \)-mixing if for any \( f_1, f_2 \in L^2(X, \nu) \),

\[
\int_G |\langle f_1, g \cdot a f_2 \rangle - \langle P_a(f_1), P_a(f_2) \rangle| \leq \varepsilon \| f_1 \|_{L^2} \| f_2 \|_{L^2}.
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As before, call the probability group $G$ itself $\varepsilon$-mixing if so are all of its measure-preserving actions.
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(Bergelson–Tao; Gowers) $D$-quasirandom groups are $D^{-1/2}$-mixing.
Quasirandom $\Rightarrow$ approximately mixing

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- (Bergelson–Tao; Gowers) $D$-quasirandom groups are $D^{-1/2}$-mixing.

**Definition**

An ultra quasirandom group $G$ is an ultraproduct $G = \prod_{n \to \alpha} G_n$ of $D_n$-quasirandom groups $G_n$ such that $D_n \to \infty$ as $n \to \infty$. 
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The ultraproduct washes the error away, so

**Proposition**

*Ultra quasirandom groups are mixing probability groups.*
Double recurrence in quasirandom groups

**Theorem (Bergelson–Tao)**

Let \((G, \mu)\) be an ultra quasirandom group and consider the left translation \(G \acts^\ell (G, \mu)\) and conjugation \(G \acts^c (G, \mu)\) actions. For any \(f_1, f_2, f_3 \in L^\infty(G, \mu)\),

\[ \forall \mu g \in G \int_G f_1(g \cdot \ell f_2)(g \cdot c f_3) = \int_G f_1 P_\ell(f_2) P_c(f_3). \]
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\]

where \(\rho(D)\) depends only on \(D\) and \(\rho(D) \to 0\) as \(D \to \infty\).
Double recurrence in quasirandom groups with explicit error

**Theorem (Bergelson–Tao)**

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Shortly after the latter result was posted, Tim Austin proved it with an explicit error \(\rho(D) = 4D^{-1/8}\), and his proof was significantly shorter than that of Bergelson–Tao.
Double recurrence in mixing probability groups

Our abstract measure theoretic approach (different from Bergelson–Tao and Austin) gives the following more general results:

**Theorem (Ts.)**

Let \((G, \mu)\) be a mixing probability group. For any \(f_1, f_2, f_3 \in L^\infty(G, \mu)\),

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Using the same proof, but keeping track of the epsilons, we get:

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Let \((G, \mu)\) be an \(\varepsilon\)-mixing probability group. For any \(f_1, f_2, f_3 \in L^2(G, \mu)\) with \(\|f_1\|_{L^\infty}, \|f_2\|_{L^\infty}, \|f_3\|_{L^\infty} \leq 1\),

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\]

Since \(D\)-quasirandom groups are \(\varepsilon = D^{-1}/2\)-mixing, we get double recurrence for them with error \(4D^{-1}/4\), which is a slight improvement over Austin's \(4D^{-1}/8\).
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THANK YOU
Proof of “mixing $\implies$ double recurrence”

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Reduction:
Writing $f_3 = (f_3 - P_c(f_3)) + P_c(f_3)$ reduces to the following two orthogonal cases:

**Case 1:** $P_c(f_3) = f_3$, i.e. $f_3$ is already conjugation-invariant. In this case, what we need to prove is $(\forall \mu g \in G) \int_G (f_1 f_3)(g \cdot \ell f_2) d\mu = \int_G (f_1 f_3) P_\ell(f_2) d\mu$, but this follows from mixing of the left translation action.

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The van der Corput trick

So we have a sequence \((e_g)_{g \in G}\) in a Hilbert space \(\mathcal{H} = L^2(G, \mu)\) and we need to understand when do we have \((\forall \mu g) \langle f, e_g \rangle = 0\), for every \(f \in \mathcal{H}\).
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Bessel's inequality implies:

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Let \((e_n)_{n \in \mathbb{N}}\) be a bounded orthogonal family in a Hilbert space \(\mathcal{H}\),
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### Random van der Corput

Let \((G, \mu)\) be a probability group and let \((e_g)_{g \in G}\) be a bounded family in a Hilbert space \(H\) such that the map \((g, h) \mapsto \langle e_g, e_h \rangle\) is \(\mathcal{B}^{(2)}\)-measurable.

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\[\forall \mu h \forall \mu g \langle e_g, e_{gh} \rangle = 0 \implies (\forall \mu g) \langle f, e_g \rangle = 0, \forall f \in H.\]

**Proof.** Fubini + a Ramsey argument reduces to the baby case. \(\square\)
Case 2: $P_c(f_3) = 0$. Goal: $\forall \mu h \forall \mu g \langle e_g, e_{gh} \rangle = 0$.

$$\langle e_g, e_{gh} \rangle = \int_G (g \cdot \ell f_2)(g \cdot c f_3)((gh) \cdot \ell f_2)((gh) \cdot c f_3)dx$$
Case 2: $P_c(f_3) = 0$. Goal: $\forall \mu h \forall \mu g \langle e_g, e_{gh} \rangle = 0$.

\[
\langle e_g, e_{gh} \rangle = \int_G (g \cdot \ell f_2)(g \cdot c f_3)((gh) \cdot \ell f_2)((gh) \cdot c f_3)dx
\]

[regrouping] $\Rightarrow \langle g \cdot \ell f_2(h \cdot \ell f_2), g \cdot c f_3(h \cdot c f_3) \rangle, \quad$
Case 2: $P_c(f_3) = 0$. Goal: $\forall \mu h \forall \mu g \langle e_g, e_{gh} \rangle = 0.$

$$
\langle e_g, e_{gh} \rangle = \int_G (g \cdot \ell f_2)(g \cdot c f_3)((gh) \cdot \ell f_2)((gh) \cdot c f_3)dx
$$

[regrouping] = $\langle g \cdot \ell f_2(h \cdot \ell f_2), g \cdot c f_3(h \cdot c f_3) \rangle,$

$[g_c = g_\ell \circ g_r] = \langle g \cdot \ell f_2(h \cdot \ell f_2), g \cdot \ell g \cdot r f_3(h \cdot c f_3) \rangle,$
Case 2: $P_c(f_3) = 0$. Goal: $\forall \mu h \forall \mu g \langle e_g, e_{gh} \rangle = 0$.

$$\langle e_g, e_{gh} \rangle = \int_G (g \cdot \ell f_2)(g \cdot c f_3)((gh) \cdot \ell f_2)((gh) \cdot c f_3)dx$$

[regrouping] = $\langle g \cdot \ell f_2(h \cdot \ell f_2), g \cdot c f_3(h \cdot c f_3) \rangle$,

$[g_c = g_\ell \circ g_r] = \langle g \cdot \ell f_2(h \cdot \ell f_2), g \cdot \ell g \cdot r f_3(h \cdot c f_3) \rangle$,

[cancellation] = $\langle f_2(h \cdot \ell f_2), g \cdot r f_3(h \cdot c f_3) \rangle$, 
Case 2: \( P_c(f_3) = 0 \). Goal: \( \forall \mu h \forall \mu g \langle e_g, e_{gh} \rangle = 0 \).

\[
\langle e_g, e_{gh} \rangle = \int_G (g \cdot \ell f_2)(g \cdot c f_3)((gh) \cdot \ell f_2)((gh) \cdot c f_3)dx
\]

[regrouping] = \( \langle g \cdot \ell f_2(h \cdot \ell f_2), g \cdot c f_3(h \cdot c f_3) \rangle \),

\( [g_c = g_\ell \circ g_r] = \langle g\cdot \ell f_2(h \cdot \ell f_2), g\cdot \ell g \cdot r f_3(h \cdot c f_3) \rangle \),

[cancellation] = \( \langle f_2(h \cdot \ell f_2), g \cdot r f_3(h \cdot c f_3) \rangle \),

\[
\text{mixing of right translation} \quad \Rightarrow \quad \forall h \forall \mu g \quad = \quad \int_G f_2(h \cdot \ell f_2) d\mu \int_G f_3(h \cdot c f_3) d\mu,
\]
Case 2: $P_c(f_3) = 0$. Goal: $\forall \mu h \forall \mu g \langle e_g, e_{gh} \rangle = 0$.

\[ \langle e_g, e_{gh} \rangle = \int_G (g \cdot \ell f_2)(g \cdot c f_3)((gh) \cdot \ell f_2)((gh) \cdot c f_3)dx \]

[regrouping] = $\langle g \cdot \ell f_2(h \cdot \ell f_2), g \cdot c f_3(h \cdot c f_3) \rangle,$

$[g_c = g_\ell \circ g_r] = \langle g \cdot \ell f_2(h \cdot \ell f_2), g \cdot \ell g \cdot r f_3(h \cdot c f_3) \rangle,$

[cancellation] = $\langle f_2(h \cdot \ell f_2), g \cdot r f_3(h \cdot c f_3) \rangle,$

\[
\begin{bmatrix}
\text{mixing of right translation} \\
\text{translation} \implies \forall h \forall \mu g
\end{bmatrix} = \int_G f_2(h \cdot \ell f_2)d\mu \int_G f_3(h \cdot c f_3)d\mu.
\]
Case 2: $P_c(f_3) = 0$. Goal: $\forall \mu h \forall \mu g \langle e_g, e_{gh} \rangle = 0$.

\[
\langle e_g, e_{gh} \rangle = \int_G (g \cdot \ell f_2)(g \cdot c f_3)((gh) \cdot \ell f_2)((gh) \cdot c f_3) dx
\]

[regrouping] $= \langle g \cdot \ell f_2(h \cdot \ell f_2), g \cdot c f_3(h \cdot c f_3) \rangle$,

$[g_c = g_{\ell} \circ g_r] = \langle g \cdot \ell f_2(h \cdot \ell f_2), g \cdot \ell g \cdot r f_3(h \cdot c f_3) \rangle$,

[cancellation] $= \langle f_2(h \cdot \ell f_2), g \cdot r f_3(h \cdot c f_3) \rangle$,

\[
\begin{bmatrix}
\text{mixing of right translation} \\
\rightarrow \forall h \forall \mu g
\end{bmatrix} = \int_G f_2(h \cdot \ell f_2) d\mu \int_G f_3(h \cdot c f_3) d\mu,
\]

\[
\begin{bmatrix}
\text{mixing of conjugation} \\
\rightarrow \forall \mu h
\end{bmatrix} = \int_G f_2(h \cdot \ell f_2) d\mu \int_G f_3 P_c(f_3) d\mu
\]
Case 2: \( P_c(f_3) = 0 \). Goal: \( \forall \mu h \forall \mu g \langle e_g, e_{gh} \rangle = 0 \).

\[
\langle e_g, e_{gh} \rangle = \int_G (g \cdot \ell f_2)(g \cdot c f_3)((gh) \cdot \ell f_2)((gh) \cdot c f_3)dx
\]

[regrouping] = \( \langle g \cdot \ell f_2(h \cdot \ell f_2), g \cdot c f_3(h \cdot c f_3) \rangle \),

[g_c = g \ell \circ g_r] = \langle g \cdot \ell f_2(h \cdot \ell f_2), g \cdot \ell g \cdot r f_3(h \cdot c f_3) \rangle ,

[cancellation] = \langle f_2(h \cdot \ell f_2), g \cdot r f_3(h \cdot c f_3) \rangle ,

\[
\begin{bmatrix}
\text{mixing of right translation} \\
\Rightarrow \forall h \forall \mu g
\end{bmatrix} = \int_G f_2(h \cdot \ell f_2) d\mu \int_G f_3(h \cdot c f_3) d\mu ,
\]

\[
\begin{bmatrix}
\text{mixing of conjugation} \\
\Rightarrow \forall \mu h
\end{bmatrix} = \int_G f_2(h \cdot \ell f_2) d\mu \int_G f_3 P_c(f_3) d\mu = 0 . \quad \Box
\]
THANK YOU