Introduction to Analytic Number Theory

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Chapter 4

Arithmetic functions III: Dirichlet series and Euler products

4.1 Introduction

Given an arithmetic function $f(n)$, the series

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \tag{4.1}$$

is called the *Dirichlet series* associated with f. A Dirichlet series can be regarded as a purely formal infinite series (i.e., ignoring questions about convergence), or as a function of the complex variable s, defined in the region in which the series converges. The variable s is usually written as

$$s = \sigma + it, \quad \sigma = \operatorname{Re} s, \quad t = \operatorname{Im} s. \tag{4.2}$$

Dirichlet series serve as a type of generating functions for arithmetic functions, adapted to the multiplicative structure of the integers, and they play a role similar to that of ordinary generating functions in combinatorics. For example, just as ordinary generating functions can be used to prove combinatorial identities, Dirichlet series can be applied to discover and prove identities among arithmetic functions.

On a more sophisticated level, the analytic properties of a Dirichlet series, regarded as a function of the complex variable s, can be exploited to obtain information on the behavior of partial sums $\sum_{n\le x} f(n)$ of arithmetic functions. This is how Hadamard and de la Vallée Poussin obtained the first proof of the Prime Number Theorem. In fact, most analytic proofs of the Prime Number Theorem (including the one we shall give in the following chapter) proceed by relating the partial sums $\sum_{n\le x}\Lambda(n)$ to a complex integral involving the Dirichlet series $\sum_{n=1}^\infty \Lambda(n)n^{-s}$, and evaluating that integral by analytic techniques.

The most famous Dirichlet series is the *Riemann zeta function* $\zeta(s)$, defined as the Dirichlet series associated with the constant function 1, i.e.,

$$\zeta(s)=\sum_{n=1}^\infty \frac{1}{n^s} \quad (\sigma>1), \tag{4.3}$$

where σ is the real part of s, as defined in (4.2).

4.2 Algebraic properties of Dirichlet series

We begin by proving two important elementary results which show that Dirichlet series "respect" the multiplicative structure of the integers. It is because of these results that Dirichlet series, rather than ordinary generating functions, are the ideal tool to study the behavior of arithmetic functions.

The first result shows that the Dirichlet series of a convolution product of arithmetic functions is the (ordinary) product of the associated Dirichlet series. It is analogous to the well-known (and easy to prove) fact that, given two functions $f(n)$ and $g(n)$, the product of their *ordinary* generating functions $\sum_{n=0}^\infty f(n)z^n$ and $\sum_{n=0}^\infty g(n)z^n$ is the generating function for the function $h(n)=\sum_{k=0}^n f(k)g(n-k)$, the *additive* convolution of f and g.

Theorem 4.1 (Dirichlet series of convolution products). *Let f and g be arithmetic functions with associated Dirichlet series $F(s)$ and $G(s)$. Let $h=f*g$ be the Dirichlet convolution of f and g, and $H(s)$ the associated Dirichlet series. If $F(s)$ and $G(s)$ converge absolutely at some point s, then so does $H(s)$, and we have $H(s)=F(s)G(s)$.*

Proof. We have

$$\begin{aligned} F(s)G(s) &= \sum_{k=1}^\infty\sum_{m=1}^\infty \frac{f(k)g(m)}{k^s m^s} \\ &= \sum_{n=1}^\infty \frac{1}{n^s}\sum_{km=n} f(k)g(m) = \sum_{n=1}^\infty \frac{(f*g)(n)}{n^s}, \end{aligned}$$

where the rearranging of terms in the double sum is justified by the absolute convergence of the series $F(s)$ and $G(s)$. This shows that $F(s)G(s) = H(s)$; the absolute convergence of the series $H(s) = \sum_{n=1}^{\infty} h(n)n^{-s}$ follows from that of $F(s)$ and $G(s)$ in view of the inequality

$$\sum_{n=1}^{\infty} \left| \frac{h(n)}{n^s} \right| \le \sum_{n=1}^{\infty} \frac{1}{|n^s|} \sum_{km=n} |f(k)| \cdot |g(m)|$$
$$= \left(\sum_{k=1}^{\infty} \left| \frac{f(k)}{k^s} \right| \right) \left(\sum_{m=1}^{\infty} \left| \frac{g(m)}{m^s} \right| \right). \qquad \square$$

Remark. The hypothesis that the Dirichlet series $F(s)$ and $G(s)$ converge *absolutely* is essential here, since one has to be able to rearrange the terms in the double series obtained by multiplying the series $F(s)$ and $G(s)$. Without this hypothesis, the conclusion of the theorem need not hold.

Corollary 4.2 (Dirichlet series of convolution inverses)**.** *Let f be an arithmetic function with associated Dirichlet series $F(s)$, and g the convolution inverse of f (so that $f*g = e$), and let $G(s)$ be the Dirichlet series associated with g. Then we have $G(s) = 1/F(s)$ at any point s at which both $F(s)$ and $G(s)$ converge absolutely.*

Proof. Since the function e has Dirichlet series $\sum_{n=1}^{\infty} e(n)n^{-s} = 1$, the result follows immediately from the theorem. $\square$

Remark. The absolute convergence of $F(s)$ does not imply that of the Dirichlet series associated with the Dirichlet inverse of f. For example, the function defined by $f(1) = 1$, $f(2) = -1$, and $f(n) = 0$ for $n \ge 3$ has Dirichlet series $F(s) = 1 - 2^{-s}$, which converges everywhere. However, the Dirichlet series of the Dirichlet inverse of f is $1/F(s) = (1 - 2^{-s})^{-1} = \sum_{k=0}^{\infty} 2^{-ks}$, which converges absolutely in $\sigma > 0$, but not in the half-plane $\sigma \le 0$.

The theorem and its corollary can be used, in conjunction with known convolution identities, to evaluate the Dirichlet series of many familiar arithmetic functions, as is illustrated by the following examples.

Examples of Dirichlet series

(1) **Unit function.** The Dirichlet series for $e(n)$, the convolution unit, is $\sum_{n=1}^{\infty} e(n)n^{-s} = 1$.

(2) **Moebius function.** Since μ is the convolution inverse of the function 1 and the associated Dirichlet series $\sum_{n=1}^{\infty} \mu(n)n^{-s}$ and $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ both converge absolutely in $\sigma > 1$, we have $\sum_{n=1}^{\infty} \mu(n)n^{-s} = 1/\zeta(s)$ for $\sigma > 1$. In particular, setting $s = 2$, we obtain the relation $\sum_{n=1}^{\infty} \mu(n)n^{-2} = 1/\zeta(2) = 6/\pi^2$, which we had derived earlier.

(3) **Characteristic function of the squares.** Let $s(n)$ denote the characteristic function of the squares. Then the associated Dirichlet series is given by $\sum_{n=1}^{\infty} s(n)n^{-s} = \sum_{m=1}^{\infty} (m^2)^{-s} = \zeta(2s)$, which converges absolutely in $\sigma > 1/2$.

(4) **Logarithm.** Termwise differentiation of the series $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ gives the series $-\sum_{n=1}^{\infty} (\log n)n^{-s}$. Since $\zeta(s)$ converges absolutely and uniformly in any range of the form $\sigma \geq 1+\epsilon$ with $\epsilon > 0$ (which follows, for example, by applying the Weierstrass M-test since the terms of the series are bounded by $n^{-1-\epsilon}$ in that range and $\sum_{n=1}^{\infty} n^{-1-\epsilon}$ converges), termwise differentiation is justified in the range $\sigma > 1$, and we therefore have $\zeta'(s) = -\sum_{n=1}^{\infty} (\log n)n^{-s}$. Hence the Dirichlet series for the function $\log n$ is $-\zeta'(s)$ and converges absolutely in $\sigma > 1$.

(5) **Identity function.** The Dirichlet series associated with the identity function is $\sum_{n=1}^{\infty} \mathrm{id}(n)n^{-s} = \sum_{n=1}^{\infty} n^{-(s-1)} = \zeta(s-1)$, which converges absolutely in $\sigma > 2$.

(6) **Euler phi function.** By the identity $\phi = \mathrm{id} * \mu$ and the formulas for the Dirichlet series for id and μ obtained above, the Dirichlet series for $\phi(n)$ is $\sum_{n=1}^{\infty} \phi(n)n^{-s} = \zeta(s-1)/\zeta(s)$ and converges absolutely for $\sigma > 2$.

(7) **Divisor function.** Since $d = 1 * 1$, the Dirichlet series for the divisor function is $\sum_{n=1}^{\infty} d(n)n^{-s} = \zeta(s)^2$ and converges absolutely in $\sigma > 1$.

(8) **Characteristic function of the squarefree numbers.** The function μ^2 satisfies the identity $\mu^2 * s = 1$, where s is the characteristic function of the squares, whose Dirichlet series was evaluated above as $\zeta(2s)$. Hence the Dirichlet series associated with μ^2, i.e., $F(s) = \sum_{n=1}^{\infty} \mu^2(n)n^{-s}$, satisfies $F(s)\zeta(2s) = \zeta(s)$, where all series converge absolutely in $\sigma > 1$. It follows that $F(s) = \zeta(s)/\zeta(2s)$ for $\sigma > 1$.

(9) **Von Mangoldt function.** Since $\Lambda * 1 = \log$ and the function log has Dirichlet series $-\zeta'(s)$ (see above), we have $\sum_{n=1}^{\infty} \Lambda(n)n^{-s}\zeta(s) =$

$-\zeta'(s)$, and so $\sum_{n=1}^{\infty}\Lambda(n)n^{-s} = -\zeta'(s)/\zeta(s)$, with all series involved converging absolutely in $\sigma > 1$. Thus, the Dirichlet series for the von Mangoldt function $\Lambda(n)$ is (up to a minus sign) equal to the logarithmic derivative of the zeta function. This relation plays a crucial role in the analytic proof of the prime number theorem, and since any zero of $\zeta(s)$ generates a singularity of the function $\sum_{n=1}^{\infty}\Lambda(n)n^{-s} = -\zeta'(s)/\zeta(s)$, it clearly shows the influence of the location of zeta zeros on the distribution of prime numbers.

The second important result of this section gives a representation of the Dirichlet series of a *multiplicative* function as an infinite product over primes, called "Euler product". Given a Dirichlet series $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$, the **Euler product** for $F(s)$ is the infinite product

$$\prod_p \left(1 + \sum_{m=1}^{\infty} \frac{f(p^m)}{p^{ms}}\right). \tag{4.4}$$

(For the definition of convergence and absolute convergence of infinite products, and some basic results about such products, see Section A.2 in the Appendix.)

Theorem 4.3 (Euler product identity)**.** *Let f be a multiplicative arithmetic function with Dirichlet series F, and let s be a complex number.*

(i) If $F(s)$ converges absolutely at some point s, then the infinite product (4.4) converges absolutely and is equal to $F(s)$.

(ii) The Dirichlet series $F(s)$ converges absolutely if and only if

$$\sum_{p^m} \left|\frac{f(p^m)}{p^{ms}}\right| < \infty. \tag{4.5}$$

Proof. (i) The absolute convergence of the infinite product follows from the bound

$$\sum_p \left|\sum_{m=1}^{\infty} \frac{f(p^m)}{p^{ms}}\right| \le \sum_p \sum_{m=1}^{\infty} \left|\frac{f(p^m)}{p^{ms}}\right| \le \sum_{n=1}^{\infty}\left|\frac{f(n)}{n^s}\right| < \infty,$$

and a general convergence criterion for infinite products (Lemma A.3 in the Appendix). It therefore remains to show that the product is equal to $F(s)$, i.e., that $\lim_{N\to\infty} P_N(s) = F(s)$, where

$$P_N(s) = \prod_{p\le N}\left(1 + \sum_{m=1}^{\infty}\frac{f(p^m)}{p^{ms}}\right).$$

Let $N \geq 2$ be given, and let $p_1, \dots, p_k$ denote the primes $\leq N$. Upon multiplying out $P_N(s)$ (note that the term 1 in each factor can be written as $f(p^m)/p^{ms}$ with $m = 0$) and using the multiplicativity of f, we obtain

$$\begin{aligned} P_N(s) &= \sum_{m_1=0}^{\infty} \cdots \sum_{m_k=0}^{\infty} \frac{f(p_1^{m_1}) \cdots f(p_k^{m_k})}{p_1^{m_1 s} \cdots p_k^{m_k s}} \\ &= \sum_{m_1=0}^{\infty} \cdots \sum_{m_k=0}^{\infty} \frac{f(p_1^{m_1} \cdots p_k^{m_k})}{(p_1^{m_1} \cdots p_k^{m_k})^s}. \end{aligned}$$

The integers $p_1^{m_1} \dots p_k^{m_k}$ occurring in this sum are positive integers composed only of prime factors $p \leq N$, i.e., elements of the set

$$A_N = \{n \in \mathbb{N} : p | n \Rightarrow p \leq N\}.$$

Moreover, by the Fundamental Theorem of Arithmetic theorem, each element of A_N has a *unique* factorization as $p_1^{m_1} \dots p_k^{m_k}$ with $m_i \in \mathbb{N} \cup \{0\}$, and thus occurs exactly once in the above sum. Hence we have $P_N(s) = \sum_{n \in A_N} f(n) n^{-s}$. Since A_N contains all integers $\leq N$, it follows that

$$|P_N(s) - F(s)| = \left| \sum_{n \notin A_N} \frac{f(n)}{n^s} \right| \leq \sum_{n > N} \left| \frac{f(n)}{n^s} \right|,$$

which tends to zero as $N \to \infty$, in view of the absolute convergence of the series $\sum_{n=1}^{\infty} f(n) n^{-s}$. Hence $\lim_{N \to \infty} P_N(s) = F(s)$.

(ii) Since the series in (4.5) is a subseries of $\sum_{n=1}^{\infty} |f(n) n^{-s}|$, the absolute convergence of $F(s)$ implies (4.5). Conversely, if (4.5) holds, then, by Lemma A.3, the infinite product

$$\prod_p \left(1 + \sum_{m=1}^{\infty} \left| \frac{f(p^m)}{p^{ms}} \right| \right)$$

converges (absolutely). Moreover, if $P_N^*(s)$ denotes the same product, but restricted to primes $p \leq N$, then, as in the proof of part (i), we have

$$P_N^*(s) = \sum_{n \in A_N} \left| \frac{f(n)}{n^s} \right| \geq \sum_{n \leq N} \left| \frac{f(n)}{n^s} \right|.$$

Since $P_N^*(s)$ converges as $N \to \infty$, the partial sums on the right are bounded as $N \to \infty$. Thus, $F(s)$ converges absolutely. □

Remarks. As in the case of the previous theorem, the result is not valid without assuming *absolute* convergence of the Dirichlet series $F(s)$.

The theorem is usually only stated in the form (i); however, for most applications the condition stated in (ii) is easier to verify than the absolute convergence of $F(s)$.

Examples of Euler products

(1) **Riemann zeta function.** The most famous Euler product is that of the Riemann zeta function, the Dirichlet series of the arithmetic function 1:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 + \sum_{m=1}^{\infty} \frac{1}{p^{ms}}\right) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \quad (\sigma > 1).$$

Euler's proof of the infinitude of primes was based on this identity. In fact, if one could take $s = 1$ in this identity one would immediately obtain the infinitude of primes since in that case the series on the left is divergent, forcing the product on the right to have infinitely many factors. However, since the identity is only valid in $\sigma > 1$, a slightly more complicated argument is needed, by applying the identity with real $s = \sigma > 1$ and investigating the behavior of the left and right sides as $s \to 1+$. If there were only finitely many primes, then the product on the right would involve only finitely many factors, and hence would converge to the finite product $\prod_p (1 - 1/p)^{-1}$ as $s \to 1+$. On the other hand, for every N, the series on the left (with $s = \sigma > 1$) is $\geq \sum_{n \leq N} n^{-\sigma}$, which converges to $\sum_{n \leq N} n^{-1}$ as $s \to 1+$. Hence the limit of the left-hand side, as $s \to 1+$, is $\geq \sum_{n \leq N} n^{-1}$ for every fixed N and, since $\sum_{n \leq N} n^{-1} \to \infty$ as $N \to \infty$, this limit must be infinite. This contradiction proves the infinitude of primes.

(2) **Moebius function.** The Dirichlet series for the Moebius function has Euler product

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right),$$

a representation that is valid in the half-plane $\sigma > 1$. This can be seen directly, by the definition of the Euler product. Alternatively, one can argue as follows: Since the Moebius function is the Dirichlet inverse of the arithmetic function 1, its Dirichlet series is the reciprocal of the Riemann zeta function. Hence, by Lemma A.4, its Euler product consists of factors that are reciprocals of the factors of the Euler product of the zeta function.

(3) **Completely multiplicative functions.** The functions 1 and μ considered above are examples of completely multiplicative functions and their inverses. The Euler products of arbitrary completely multiplicative functions and their inverses have the same general shape. Indeed, let f be a completely multiplicative function with Dirichlet series $F(s)$, and let g be the Dirichlet inverse of f, with Dirichlet series $G(s)$. Then, formally, we have the identities

$$F(s) = \prod_p \left(\sum_{m=0}^{\infty} \frac{f(p)^m}{p^{ms}} \right) = \prod_p \left(1 - \frac{f(p)}{p^s} \right)^{-1},$$

and

$$G(s) = \frac{1}{F(s)} = \prod_p \left(1 - \frac{f(p)}{p^s} \right).$$

These representations are valid provided the associated Dirichlet series converge absolutely, a condition that can be checked, for example, using the criterion of part (ii) of Theorem 4.3. For example, if $|f(p)| \leq 1$ for all primes p, then both $F(s)$ and $G(s)$ converge absolutely in $\sigma > 1$, and so the Euler product representations are valid in $\sigma > 1$ as well.

(4) **Characteristic function of integers relatively prime to a given set of primes.** Given a finite or infinite set of primes $\mathcal{P}$, let $f_{\mathcal{P}}$ denote the characteristic function of the positive integers that do not have a prime divisor belonging to the set $\mathcal{P}$. Thus $f_{\mathcal{P}}$ is the completely multiplicative function defined by $f_{\mathcal{P}} = 1$ if $p \notin \mathcal{P}$ and $f_{\mathcal{P}} = 0$ if $p \in \mathcal{P}$. Then the Dirichlet series $F_{\mathcal{P}}$ of $f_{\mathcal{P}}$ is given by the Euler product

$$F_{\mathcal{P}}(s) = \prod_{p \notin \mathcal{P}} \left(1 - \frac{1}{p^s} \right)^{-1} = \zeta(s) \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p^s} \right),$$

and this representation is valid in $\sigma > 1$.

(5) **Characteristic function of k-free integers.** Given an integer $k \geq 2$, fet $f_k(n)$ denote the characteristic function of the "k-free" integers, i.e., integers which are not divisible by the k-th power of a prime. The

function f_k is obviously multiplicative, and since it is bounded by 1, its Dirichlet series $F_k(s)$ converges absolutely in the half-plane $\sigma > 1$ and there has Euler product

$$F_k(s) = \prod_p \left(\sum_{m=0}^{k-1} \frac{1}{p^{ms}} \right) = \prod_p \frac{1 - p^{-ks}}{1 - p^{-s}} = \frac{\zeta(s)}{\zeta(ks)}.$$

(6) **Euler phi function.** Since $\phi(p^m) = p^m - p^{m-1}$ for $m \geq 1$, we have, formally,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} &= \prod_p \left(1 + \sum_{m=1}^{\infty} \frac{p^m - p^{m-1}}{p^{ms}} \right) \\ &= \prod_p \left(1 + \frac{1 - p^{-1}}{p^{s-1}(1 - p^{-s+1})} \right) = \prod_p \frac{1 - p^{-s}}{1 - p^{-s+1}}.\end{aligned}$$

Since $\phi(n) \leq n$, the Dirichlet series for ϕ converges absolutely in the half-plane $\sigma > 2$, so the above Euler product representation is valid in this half-plane. Moreover, the last expression above can be recognized as the product of the Euler product representations for the Dirichlet series $\zeta(s-1)$ and $1/\zeta(s)$. Thus, the Dirichlet series for $\phi(n)$ is equal to $\zeta(s-1)/\zeta(s)$, a result we had obtained earlier using the identity $\phi = \mathrm{id} * \mu$.

4.3 Analytic properties of Dirichlet series

We begin by proving two results describing the regions in the complex plane in which a Dirichlet series converges, absolutely or conditionally.

In the case of an ordinary power series $\sum_{n=0}^{\infty} a_n z^n$, it is well-known that there exists a "disk of convergence" $|z| < R$ such that the series converges absolutely $|z| < R$, and diverges when $|z| > R$. The number R, called "radius of convergence", can be any positive real number, or 0 (in which case the series diverges for all $z \neq 0$), or ∞ (in which case the series converges everywhere). For values z on the circle $|z| = R$, the series may converge or diverge.

For Dirichlet series, a similar result is true, with the disk of convergence replaced by a half-plane of convergence of the form $\sigma > \sigma_0$. However, in contrast to the situation for power series, where the regions for convergence and absolute convergence are identical (except possibly for the boundaries), for Dirichlet series there may be a nontrivial region in the form of a vertical strip in which the series converges, but does not converge absolutely.

Theorem 4.4 (Absolute convergence of Dirichlet series). *For every Dirichlet series there exists a number $\sigma_a \in \mathbb{R} \cup \{\pm\infty\}$, called the* abscissa of absolute convergence*, such that for all s with $\sigma > \sigma_a$ the series converges absolutely, and for all s with $\sigma < \sigma_a$, the series does not converge absolutely.*

Proof. Given a Dirichlet series $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$, let A be the set of complex numbers s at which $F(s)$ converges absolutely. If the set A is empty, the conclusion of the theorem holds with $\sigma_a = \infty$. Otherwise, set $\sigma_a = \inf\{\operatorname{Re} s : s \in A\} \in \mathbb{R} \cup \{-\infty\}$. By the definition of σ_a, the series $F(s)$ does not converge absolutely if $\sigma < \sigma_a$. On the other hand, if $s = \sigma + it$ and $s' = \sigma' + it'$ with $\sigma' \geq \sigma$, then

$$\sum_{n=1}^{\infty} \left| \frac{f(n)}{n^{s'}} \right| = \sum_{n=1}^{\infty} \frac{|f(n)|}{n^{\sigma'}} \leq \sum_{n=1}^{\infty} \frac{|f(n)|}{n^{\sigma}} = \sum_{n=1}^{\infty} \left| \frac{f(n)}{n^{s}} \right|.$$

Hence, if $F(s)$ converges absolutely at some point s, then it also converges absolutely at any point s' with $\operatorname{Re} s' \geq \operatorname{Re} s$. Since, by the definition of σ_a, there exist points s with σ arbitrarily close to σ_a at which the Dirichlet series $F(s)$ converges absolutely, it follows that the series converges absolutely at *every* point s with $\sigma > \sigma_a$. This completes the proof of the theorem. □

Remark. In the case when $\sigma = \sigma_a$, the series may or may not converge absolutely. For example, the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ has abscissa of absolute convergence $\sigma_a = 1$, but it does not converge absolutely when $\sigma = 1$. On the other hand, the Dirichlet series corresponding to the arithmetic function $f(n) = 1/\log^2(2n)$ has the same abscissa of convergence 1, but also converges absolutely at $\sigma = 1$.

Establishing an analogous result for *conditional* convergence is harder. The key step is contained in the following theorem.

Proposition 4.5. *Suppose the Dirichlet series $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ converges at some point $s = s_0 = \sigma_0 + it_0$. Then the series converges at every point s with $\sigma > \sigma_0$. Moreover, the convergence is uniform in every compact region contained in the half-plane $\sigma > \sigma_0$.*

Proof. Suppose $F(s)$ converges at s_0, and let s be a point with $\sigma > \sigma_0$. Set $\delta = \sigma - \sigma_0$ (so that $\delta > 0$) and let

$$S(x, y) = \sum_{x < n \leq y} \frac{f(n)}{n^s}, \quad S_0(x, y) = \sum_{x < n \leq y} \frac{f(n)}{n^{s_0}} \quad (y > x \geq 1).$$

We will establish the convergence of the series $F(s)$ by showing that it satisfies Cauchy's criterion.

Let $\epsilon > 0$ be given. By Cauchy's criterion, applied to the series $F(s_0) = \sum_{n=1}^{\infty} f(n)n^{-s_0}$ (which, by hypothesis, converges) there exists $x_0 = x_0(\epsilon) \geq 1$ such that

$$|S_0(x,y)| \leq \epsilon \quad (y > x \geq x_0). \tag{4.6}$$

We now relate the sums $S(x,y)$ to the sums $S_0(x,y)$ by writing the summands as $f(n)n^{-s_0} \cdot n^{s_0-s}$, and "removing" the factor n^{s_0-s} by partial summation. Given $y > x \geq x_0$, we have

$$\begin{aligned} S(x,y) &= \sum_{x<n\leq y} \frac{f(n)}{n^{s_0}} \cdot n^{s_0-s} \\ &= S_0(x,y)y^{s_0-s} - \int_x^y S_0(x,u)(s_0-s)u^{s_0-s-1}du. \end{aligned}$$

Since, by (4.6), $|S_0(x,u)| \leq \epsilon$ for $u \geq x(\geq x_0)$, and $|u^{s_0-s}| = u^{\sigma_0-\sigma} = u^{-\delta}$ with $\delta > 0$, we obtain

$$\begin{aligned} |S(x,y)| &\leq \epsilon y^{-\delta} + \epsilon|s-s_0| \int_x^y u^{-\delta-1}du \\ &\leq \epsilon \left(1 + |s-s_0| \int_1^{\infty} u^{-\delta-1}du\right) \\ &= \epsilon \left(1 + \frac{|s-s_0|}{\delta}\right) = C\epsilon \quad (y > x \geq x_0(\epsilon)), \end{aligned}$$

where $C = C(s,s_0) = 1 + |s-s_0|/\delta$ is independent of x and y. Since ϵ was arbitrary, this shows that the series $F(s)$ satisfies Cauchy's criterion and hence converges.

To prove that the convergence is uniform on compact subsets of the half plane $\sigma > \sigma_0$, note that in any compact subset K of the half-plane $\sigma > \sigma_0$, the quantity $\delta = \sigma - \sigma_0$ is bounded from below and $|s-s_0|$ is bounded from above. Hence, the constant $C = C(s,s_0) = |s-s_0|/\delta$ defined above is bounded by a constant $C_0 = C_0(K)$ depending only on the subset K, but not on s, and the Cauchy criterion therefore holds uniformly in K. □

The following result describes the region of convergence of a Dirichlet series and is the analog of Theorem 4.4 for conditional convergence.

Theorem 4.6 (Convergence of Dirichlet series). *For every Dirichlet series there exists a number* $\sigma_c \in \mathbb{R} \cup \{\pm\infty\}$*, called the* abscissa of convergence*,* *such that the series converges in the half-plane $\sigma > \sigma_c$ (the "half-plane of convergence"), and diverges in the half-plane $\sigma < \sigma_c$. The convergence is uniform on compact subsets of the half-plane of convergence. Moreover, the abscissa of convergence σ_c and the abscissa of absolute convergence σ_a satisfy $\sigma_a - 1 \le \sigma_c \le \sigma_a$.*

Remarks. As in the case of Theorem 4.4, at points s on the line $\sigma = \sigma_c$, the series may converge or diverge.

The inequalities $\sigma_a - 1 \le \sigma_c \le \sigma_a$ are best-possible, in the sense that equality can occur in both cases, as illustrated by the following examples.

(i) If $f(n)$ is nonnegative, then, at any point s *on the real line* the associated Dirichlet series $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ converges if and only if it converges absolutely. Since, by the theorem, convergence (and absolute convergence) occurs on half-planes, this implies that the half-planes of convergence and absolute convergence are identical. Hence we have $\sigma_c = \sigma_a$ whenever $f(n)$ is nonnegative.

(ii) The Dirichlet series $F(s) = \sum_{n=1}^{\infty}(-1)^n n^{-s}$ is an example for which $\sigma_c = \sigma_a - 1$. Indeed, $F(s)$ converges at any *real* s with $s > 0$ (since it is an alternating series with decreasing terms at such points), and diverges for $\sigma \le 0$ (since for $\sigma \le 0$ the terms of the series do not converge to zero), so we have $\sigma_c = 0$. However, since $\sum_{n=1}^{\infty} |(-1)^n n^{-s}| = \sum_{n=1}^{\infty} n^{-\sigma}$, the series converges absolutely if and only if $\sigma > 1$, so that $\sigma_a = 1$.

Proof. If the series $F(s)$ diverges everywhere, the result holds with $\sigma_c = \infty$. Suppose therefore that the series converges at at least one point, let D be the set of all points s at which the series converges, and define $\sigma_c = \inf\{\operatorname{Re} s : s \in D\} \in \mathbb{R} \cup \{-\infty\}$. Then, by the definition of σ_c, $F(s)$ diverges at any point s with $\sigma < \sigma_c$. On the other hand, there exist points $s_0 = \sigma_0 + it_0$ with σ_0 arbitrarily close to σ_c such that the series converges at s_0. By Proposition 4.5 it follows that, given such a point s_0, the series $F(s)$ converges at every point s with $\sigma > \sigma_0$, and the convergence is uniform in compact subsets of $\sigma > \sigma_0$. Since σ_0 can be taken arbitrarily close to σ_c, it follows that $F(s)$ converges in the half-plane $\sigma > \sigma_c$, and that the convergence is uniform on compact subsets of this half-plane.

To obtain the last assertion of the theorem, note first that the inequality $\sigma_c \le \sigma_a$ holds trivially since absolute convergence implies convergence. Moreover, if $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ converges at some point $s = s_0 = \sigma_0 + it_0$, then $f(n)n^{-s_0}$ tends to zero as $n \to \infty$, so that, in particular, $|f(n)n^{-s_0}| \le 1$ for $n \ge n_0$, say. Hence, for $n \ge n_0$ and any s we have $|f(n)n^{-s}| \le n^{-(\sigma-\sigma_0)}$, and since the series $\sum_{n=1}^{\infty} n^{-(\sigma-\sigma_0)}$ converges whenever $\sigma > \sigma_0 + 1$, it follows that $F(s)$ converges absolutely in

$\sigma > \sigma_0 + 1$. Since σ_0 can be taken arbitrarily close to σ_c, this implies that $\sigma_a \le \sigma_c + 1$. □

We are now ready to prove the most important result about Dirichlet series, namely that Dirichlet series are analytic functions of s in their half-plane of convergence. It is this result that allows one to apply the powerful apparatus of complex analysis to the study of arithmetic functions.

Theorem 4.7 (Analytic properties of Dirichlet series)**.** *A Dirichlet series $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ represents an analytic function of s in its half-plane of convergence $\sigma > \sigma_c$. Moreover, in the half-plane of convergence, the Dirichlet series can be differentiated termwise, that is, we have $F'(s) = -\sum_{n=1}^{\infty} f(n)(\log n)n^{-s}$, and the latter series also converges in this half-plane.*

Proof. Let $F_N(s) = \sum_{n=1}^{N} f(n)n^{-s}$ denote the partial sums of $F(s)$. Since each term $f(n)n^{-s} = f(n)e^{-s(\log n)}$ is an entire function of s, the functions $F_N(s)$ are entire. By Theorem 4.6, as $N \to \infty$, $F_N(s)$ converges to $F(s)$, uniformly on compact subsets of the half-plane $\sigma > \sigma_c$. By Weierstrass' theorem on uniformly convergent sequences of analytic functions, it follows that $F(s)$ is analytic in every compact subset of the half-plane $\sigma > \sigma_c$, and hence in the entire half-plane. This proves the first assertion of the theorem. The second assertion regarding termwise differentiation follows since the finite partial sums $F_N(s)$ can be termwise differentiated with derivative $F_N'(s) = \sum_{n=1}^{N} f(n)(-\log n)n^{-s}$, and since, by another application of Weierstrass' theorem, the derivatives $F_N'(s)$ converge to $F'(s)$ in the half-plane $\sigma > \sigma_c$. □

Remark. Note that the analyticity of $F(s)$ holds in the half-plane of convergence $\sigma > \sigma_c$, not just in the (smaller) half-plane $\sigma > \sigma_a$ in which the series converges *absolutely.*

The next theorem is a simple, but very useful result, which shows that an arithmetic function is uniquely determined by its Dirichlet series.

Theorem 4.8 (Uniqueness theorem for Dirichlet series)**.** *Suppose $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ and $G(s) = \sum_{n=1}^{\infty} g(n)n^{-s}$ are Dirichlet series with finite abscissa of convergence that satisfy $F(s) = G(s)$ for all s with σ sufficiently large. Then $f(n) = g(n)$ for all n.*

Proof. Set $h(n) = f(n) - g(n)$ and let $H(s) = F(s) - G(s)$ be the Dirichlet series for h. By the hypotheses of the theorem there exists σ_0 such that $H(s)$ converges absolutely in the half-plane $\sigma \ge \sigma_0$, and is identically 0 in this half-plane. We need to show that $h(n) = 0$ for all n. To get a contradiction, suppose h is not identically 0, and let n_0 be the smallest positive integer n such that $h(n) \neq 0$. Then $H(s) = h(n_0)n_0^{-s} + H_1(s)$, where $H_1(s) = \sum_{n=n_0+1}^{\infty} h(n)n^{-s}$. Since $H(s) = 0$ for $\sigma \geq \sigma_0$, it follows that, for any $\sigma \geq \sigma_0$, $h(n_0)n_0^{-\sigma} = -H_1(\sigma)$, and hence

$$|h(n_0)| \leq |H_1(\sigma)|n_0^{\sigma} \leq \sum_{n=n_0+1}^{\infty} |h(n)|\frac{n_0^{\sigma}}{n^{\sigma}}.$$

Setting $\sigma = \sigma_0 + \lambda$ with $\lambda \geq 0$, we have, for $n \geq n_0 + 1$,

$$\frac{n_0^{\sigma}}{n^{\sigma}} = \left(\frac{n_0}{n}\right)^{\lambda}\left(\frac{n_0^{\sigma_0}}{n^{\sigma_0}}\right) \leq \left(\frac{n_0}{n_0+1}\right)^{\lambda}\left(\frac{n_0^{\sigma_0}}{n^{\sigma_0}}\right),$$

so that

$$|h(n_0)| \leq n_0^{\sigma_0}\left(\frac{n_0}{n_0+1}\right)^{\lambda} \sum_{n=n_0+1}^{\infty} \frac{|h(n)|}{n^{\sigma_0}} = C_0\left(\frac{n_0}{n_0+1}\right)^{\lambda},$$

where $C_0 = n_0^{\sigma_0}\sum_{n=n_0+1}^{\infty} |h(n)|n^{-\sigma_0}$ is a (finite) constant, independent of λ, by the absolute convergence of $H(\sigma_0)$. Letting $\lambda \to \infty$, the right-hand side tends to zero, contradicting our hypothesis that $h(n_0) \neq 0$. □

Corollary 4.9 (Computing convolution inverses via Dirichlet series)**.** *Let $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ be a Dirichlet series with finite abscissa of convergence, and suppose that $1/F(s) = G(s)$, where $G(s) = \sum_{n=1}^{\infty} g(n)n^{-s}$ is a Dirichlet series with finite abscissa of convergence. Then g is the convolution inverse of f.*

Proof. Let $h = f * g$, and let $H(s)$ be the Dirichlet series for h. By the hypotheses of the corollary and Theorems 4.4 and 4.6, the series $F(s)$ and $G(s)$ converge absolutely in a half-plane $\sigma \geq \sigma_0$. By Theorem 4.1 it then follows that $H(s)$ also converges absolutely in the same half-plane and is equal to $F(s)G(s)$ there. On the other hand, since $G(s) = 1/F(s)$, we have $H(s) = 1 = \sum_{n=1}^{\infty} e(n)n^{-s}$. By the uniqueness theorem it follows that $h(n) = e(n)$ for all n, i.e., we have $h = f * g = e$. □

Application: Proving identities for arithmetic functions via Dirichlet series. The uniqueness theorem and its corollary provide a new method for obtaining identities among arithmetic functions and computing convolution inverses. In order to prove an identity of the form $f(n) \equiv g(n)$, it suffices to show that the corresponding Dirichlet series converge and are equal for sufficiently large σ. In practice, this is usually carried out by algebraically manipulating the Dirichlet series for $f(n)$ to obtain another Dirichlet series and then "reading off" the coefficients of the latter Dirichlet series, to conclude that these coefficients must be equal to those in the original Dirichlet series. In most cases, the functions involved are multiplicative, so that the Dirichlet series can be written as Euler products, and it is the individual factors in the Euler product that are manipulated. We illustrate this technique with the following examples.

Examples

(1) **Alternate proof of the identity $\phi = \mu * \mathrm{id}$.** Using only the multiplicativity of ϕ and the definition of ϕ at prime powers, we have shown above that the Dirichlet series of ϕ is equal to $\zeta(s-1)/\zeta(s)$. Since $\zeta(s-1)$ and $1/\zeta(s)$ are, respectively, the Dirichlet series of the functions id and μ, $\zeta(s-1)/\zeta(s)$ is the Dirichlet series of the convolution product $\mathrm{id} * \mu$. Since both of these series converge absolutely for $\sigma > 2$, we can apply the uniqueness theorem for Dirichlet series to conclude that $\phi = \mathrm{id} * \mu$.

(2) **Computing "square roots" of completely multiplicative functions.** Given a completely multiplicative function f, we want to find a function g such that $g * g = f$. To this end note that if g satisfies $g * g = f$, then the corresponding Dirichlet series $G(s)$ must satisfy $G(s)^2 = F(s)$, provided $G(s)$ converges absolutely. Thus, we seek a function whose Dirichlet series is the square root of the Dirichlet series for f. Now, since f is completely multiplicative, its Dirichlet series has an Euler product with factors of the form $(1 - f(p)p^{-s})^{-1}$. Taking the square root of this expression and using the binomial series $(1+x)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} x^n$ gives

$$\left(1 - \frac{f(p)}{p^s}\right)^{-1/2} = 1 + \sum_{m=1}^{\infty} \binom{-1/2}{m} \frac{(-1)^m f(p)^m}{p^{ms}}.$$

The latter series can be identified as the p-th factor of the Euler product of the multiplicative function g defined by $g(p^m) = (-f(p))^m \binom{-1/2}{m}$. Let $G(s)$ be the Dirichlet series for g. Then $G(s)^2 = F(s)$, and the uniqueness theorem yields $g * g = f$ provided both series $G(s)$ and $F(s)$ have finite abscissa of convergence. The bound

$|\binom{-1/2}{m}| = |(-1)^m \binom{2m}{m}| \le 2^{2m}$ shows that this convergence condition is satisfied if, for example, the values $f(p)$ are uniformly bounded.

(3) **Computing convolution inverses.** A direct computation of convolution inverses requires solving an infinite system of linear equations, but Dirichlet series often allow a quick computation of an inverse. As an application of Corollary 4.9, consider the function f defined by $f(1) = 1$, $f(2) = -1$, and $f(n) = 0$ for $n \ge 3$. This function has Dirichlet series $F(s) = 1 - 2^{-s}$, which is an entire function of s. The reciprocal of $F(s)$ is given by $1/F(s) = (1 - 2^{-s})^{-1} = \sum_{k=0}^{\infty} 2^{-ks}$. Writing this series in the form $G(s) = \sum_{n=1}^{\infty} g(n)n^{-s}$, and reading off the coefficients $g(n)$, we see that $1/F(s)$ is the Dirichlet series of the function g defined by $g(n) = 1$ if $n = 2^k$ for some nonnegative integer k, and $g(n) = 0$ otherwise. The series $G(s)$ converges absolutely for $\sigma > 0$. Hence, Corollary 4.9 is applicable and shows that g is the convolution inverse of f.

(4) **Evaluating functions via Dirichlet series.** Another type of application is illustrated by the following example. Let $f_k(n)$ denote the characteristic function of k-free integers. In order to estimate the partial sums $\sum_{n \le x} f_k(n)$ (i.e., the number of k-free positive integers $\le x$), a natural approach is to use the convolution method with the function 1 as the approximating function. This requires computing the "perturbation factor" g_k defined by $f_k = 1 * g_k$. If F_k and G_k denote the Dirichlet series of f_k and g_k, respectively, then $G_k(s) = F_k(s)/\zeta(s)$. In the previous section, we computed $F_k(s)$ as $F_k(s) = \zeta(s)/\zeta(ks)$, so

$$G_k(s) = \frac{1}{\zeta(ks)} = \prod_p \left(1 - \frac{1}{p^{ks}}\right).$$

The latter product is the Euler product of the Dirichlet series for the multiplicative function g_k^* defined by $g_k^*(p^m) = -1$ if $m = k$ and $g_k^*(p^m) = 0$ otherwise, i.e., $g_k^*(n) = \mu(n^{1/k})$ if n is a k-th power, and $g_k^*(n) = 0$ otherwise. Since all series involved converge absolutely for $\sigma > 1$, the uniqueness theorem applies, and we conclude that the coefficients of the latter series and those of $G_k(s)$ must be equal, i.e., we have $g_k \equiv g_k^*$.

(5) **Wintner's theorem for multiplicative functions.** In the terminology and notation of Dirichlet series, Wintner's theorem (Theorem 2.19) states that if $f = 1 * g$ and if the Dirichlet series $G(s)$ of g

converges absolutely at $s = 1$, then the mean value $M(f)$ exists and is equal to $G(1)$. This result holds for arbitrary arithmetic functions f and g satisfying the above conditions, but if these functions are multiplicative, then one can express the mean value $M(f)$ as an Euler product, and one can check the condition that the Dirichlet series $G(s)$ converges absolutely at $s = 1$ by the criterion of Theorem 4.3: Applying Theorem 4.3 to the function $g = f * \mu$, noting that $g(p^m) = f(p^m) - f(p^{m-1})$ for every prime power p^m, and using the fact that if f is multiplicative, then so is $g = f * \mu$, we obtain the following version of Wintner's theorem for multiplicative functions:

Suppose f is multiplicative and satisfies

$$\sum_{p^m} \frac{|f(p^m) - f(p^{m-1})|}{p^m} < \infty.$$

Then the mean value $M(f)$ exists and is given by

$$\begin{aligned} M(f) &= \prod_p \left(1 + \sum_{m=1}^{\infty} \frac{f(p^m) - f(p^{m-1})}{p^m}\right) \\ &= \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \sum_{m=1}^{\infty} \frac{f(p^m)}{p^m}\right). \end{aligned}$$

4.4 Dirichlet series and summatory functions

4.4.1 Mellin transform representation of Dirichlet series

As we have seen in Chapter 2, to investigate the behavior of arithmetic functions one usually considers the associated summatory functions

$$M(f, x) = \sum_{n \le x} f(n), \tag{4.7}$$

or weighted versions of those sums, such as the "logarithmic" sums

$$L(f, x) = \sum_{n \le x} \frac{f(n)}{n}. \tag{4.8}$$

In contrast to the individual values $f(n)$, which for most natural arithmetic functions oscillate wildly and show no discernable pattern when $n \to \infty$, the summatory functions $M(f,x)$ and $L(f,x)$ are usually well-behaved and can be estimated in a satisfactory manner. Most results and problems on arithmetic functions can be expressed in terms of these summatory functions. For example, as we have seen in Chapter 2, Mertens' estimates show that $L(\Lambda,x)=\log x+O(1)$ and the PNT is equivalent to the asymptotic formula $M(\Lambda,x)\sim x$.

It is therefore natural to try to express the Dirichlet series $F(s)$ of an arithmetic function f in terms of the summatory functions $M(f,x)$ and vice versa, to exploit this to translate between properties of the analytic function $F(s)$ those of the arithmetic quantity $M(f,x)$.

In one direction, namely going from $M(f,x)$ to $F(s)$, this is rather easy. The key result is the following theorem which expresses $F(s)$ as an integral over $M(f,x)$ and is a restatement (in a slightly different notation) of Theorem 2.15. The converse direction is considerably more difficult, and will be considered in a separate section.

Theorem 4.10 (Mellin transform representation of Dirichlet series)**.** *Let $F(s)=\sum_{n=1}^\infty f(n)n^{-s}$ be a Dirichlet series with finite abscissa of convergence σ_c, and let $M(f,x)$ and $L(f,x)$ be given by* (4.7) *and* (4.8)*, respectively. Then we have*

$$F(s)=s\int_1^\infty M(f,x)x^{-s-1}dx \quad (\sigma>\max(0,\sigma_c)), \tag{4.9}$$

$$F(s)=(s-1)\int_1^\infty L(f,x)x^{-s}dx \quad (\sigma>\max(1,\sigma_c)). \tag{4.10}$$

Proof. We first show that the second relation follows from the first, applied to the function $\tilde f(n)=f(n)/n$, with $\tilde s=s-1$ in place of s, and $\tilde F(s)=\sum_{n=1}^\infty \tilde f(n)n^{-s}$ in place of $F(s)$. To this end, observe first that $L(f,x)=M(\tilde f,x)$, so the right-hand side of (4.10) becomes the right-hand side of (4.9) with $\tilde f$ in place of f and $\tilde s=s-1$ in place of s. Moreover, we have $F(s)=\tilde F(s-1)=\tilde F(\tilde s)$, so the left-hand sides of these relations are also equal under these substitutions. Finally, since $\tilde F(s)=F(s+1)$, the abscissa of convergence $\tilde\sigma_c$ of $\tilde F$ is equal to σ_c-1, so the condition $\tilde\sigma>\max(0,\tilde\sigma_c)$ in (4.9) translates into $\sigma-1>\max(0,\sigma_c-1)$, or, equivalently, $\sigma>\max(1,\sigma_c)$, which is the condition in (4.10).

The first relation, (4.9), was already proved in Theorem 2.15 of Chapter 2, as an application of partial summation. We give here an alternate argument: Let f and $F(s)$ be given as in the theorem, and fix s with

$\sigma > \min(0, \sigma_c)$, so that the Dirichlet series $F(s)$ converges at s. Write

$$M(f,x) = \sum_{n=1}^{\infty} \chi(n,x) f(n),$$

where

$$\chi(x,n) = \begin{cases} 1 & \text{if } n \leq x, \\ 0 & \text{if } n > x. \end{cases}$$

Then, for every $X \geq 1$,

$$\begin{aligned} s\int_1^X M(f,x)x^{-s-1}dx &= s\int_1^X \sum_{n=1}^{\infty} \chi(x,n)f(n)x^{-s-1}dx \\ &= s\int_1^X \sum_{n\leq X} \chi(x,n)f(n)x^{-s-1}dx \\ &= s\sum_{n\leq X} f(n)\int_1^X \chi(x,n)x^{-s-1}dx \\ &= s\sum_{n\leq X} f(n)\int_n^X x^{-s-1}dx \\ &= s\sum_{n\leq X} f(n)\frac{1}{s}\left(\frac{1}{n^s} - \frac{1}{X^s}\right) \\ &= \sum_{n\leq X} \frac{f(n)}{n^s} - \frac{1}{X^s}M(f,X). \end{aligned}$$

Now let $X \to \infty$. Then, by the convergence of $F(s)$, the first term on the right tends to $F(s)$. Moreover, Kronecker's Lemma (Theorem 2.12) implies that the second term tends to 0. Hence we conclude

$$\lim_{X\to\infty} s\int_1^X M(f,x)x^{-s-1}dx = F(s),$$

which proves (4.9). □

Despite its rather elementary nature and easy proof, this result has a number of interesting and important applications, as we will illustrate in the following subsections.

4.4.2 Analytic continuation of the Riemann zeta function

As a first application of Theorem 4.10, we give an integral representation for the Riemann zeta function that is valid in the half-plane $\sigma > 0$ and provides an analytic continuation of $\zeta(s)$ to this half-plane.

Theorem 4.11 (Integral representation and analytic continuation of the zeta function)**.** *The Riemann zeta function, defined for $\sigma > 1$ by the series*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{4.11}$$

has an analytic continuation to a function defined on the half-plane $\sigma > 0$ and is analytic in this half-plane with the exception of a simple pole at $s = 1$ with residue 1, given by

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \{x\}x^{-s-1}dx \quad (\sigma > 0). \tag{4.12}$$

Remark. Strictly speaking, we should use a different symbol, say $\tilde{\zeta}(s)$, for the analytic continuation defined by (4.12). However, to avoid awkward notations, it has become standard practice to denote the analytic continuation of a Dirichlet series by the same symbol as the series itself, and we will usually follow this practice. That said, one should be aware that the validity of the series representation in general does not extend to the region in which the Dirichlet series is analytic (in the sense of having an analytic continuation there). For example, the Dirichlet series representation (4.11) of the zeta function diverges at every point in the half-plane $\sigma < \sigma_c = 1$ (and even at every point on the line $\sigma = 1$, as one can show by Euler's summation), and thus is not even well defined outside the half-plane $\sigma > 1$. By contrast, the representation (4.12) is well-defined in the larger half-plane $\sigma > 0$ and represents an analytic function there.

Proof. Applying Theorem 4.10 with $f \equiv 1$, $F(s) = \zeta(s)$, $\sigma_c = 1$, and $M(f, x) = [x]$, we obtain

$$\zeta(s) = s\int_1^\infty [x]x^{-s-1}dx \quad (\sigma > 1).$$

Setting $[x] = x - \{x\}$, where $\{x\}$ is the fractional part of x, we can write the last integral as

$$\int_1^\infty x^{-s}dx - \int_1^\infty \{x\}x^{-s-1}dx = \frac{1}{s-1} - \int_1^\infty \{x\}x^{-s-1}dx,$$

and thus obtain the representation (4.12) in the half-plane $\sigma > 1$. Now note that, given $\epsilon > 0$, the integral in (4.12) is bounded, for any s with $\sigma \geq \epsilon$, by

$$\left| \int_1^\infty \{x\} x^{-s-1} dx \right| \leq \int_1^\infty x^{-\sigma-1} dx \leq \int_1^\infty x^{-\epsilon-1} dx = \frac{1}{\epsilon}.$$

Hence this integral converges absolutely and uniformly in the half-plane $\sigma \geq \epsilon$ and therefore represents an analytic function of s in the half-plane $\sigma \geq \epsilon$. Since $\epsilon > 0$ can be taken arbitrarily small, this function is in fact analytic in the half-plane $\sigma > 0$. It follows that the right-hand side of (4.12) is an analytic function in this half-plane, with the exception of the pole at $s = 1$ with residue 1, coming from the term $s/(s-1)$. This provides the asserted analytic continuation of $\zeta(s)$ to the half-plane $\sigma > 0$. □

As an immediate consequence of the representation (4.12) for $\zeta(s)$, we obtain an estimate for $\zeta(s)$ near the point $s = 1$.

Corollary 4.12 (Estimate for $\zeta(s)$ near $s = 1$)**.** *For $|s-1| \leq 1/2$, $s \neq 1$, we have*

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(|s-1|),$$

where γ is Euler's constant.

Proof. By Theorem 4.11, the function $\zeta(s) - 1/(s-1)$ is analytic in the disk $|s-1| < 1$ and therefore has a power series expansion

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=0}^\infty a_n (s-1)^n$$

in this disk. It follows that

$$\zeta(s) - \frac{1}{s-1} = a_0 + O(|s-1|)$$

in the disk $|s-1| \leq 1/2$. Thus it remains to show that the constant a_0 is equal to γ. By (4.12) we have, in the half-plane $\sigma > 0$,

$$\zeta(s) - \frac{1}{s-1} = 1 - s \int_1^\infty \{x\} x^{-s-1} dx.$$

Letting $s \to 1$ on the right-hand side, we get

$$a_0 = \lim_{s \to 1} \left(\zeta(s) - \frac{1}{s-1} \right) = 1 - \int_1^\infty \{x\} x^{-2} dx.$$

Now, from the proof of the harmonic sum estimate (Theorem 2.5) we have $\gamma = 1 - \int_1^\infty \{x\} x^{-2} dx$, so we obtain $a_0 = \gamma$ as claimed. □

4.4.3 Lower bounds for error terms in summatory functions

We begin with a general result relating error terms in estimates for the summatory functions $M(f,x)$ to the region of analyticity of the Dirichlet series $F(s)$.

Theorem 4.13 (Error terms in estimates for $M(f,x)$ and analyticity of $F(s)$)**.**

(i) If $M(f,x) = O(x^\theta)$ for some $\theta \ge 0$, then $F(s)$ is analytic in the half-plane $\sigma > \theta$.

(ii) If $M(f,x) = Ax^\alpha + O(x^\theta)$ for some constants A, α and θ with $\alpha > \theta \ge 0$, then $F(s) - As(s-\alpha)^{-1}$ is analytic in the half-plane $\sigma > \theta$.

Proof. First note that since $f(n) = M(f,n) - M(f,n-1)$, the given estimates for $M(f,x)$ imply that $f(n) = O(n^\theta)$ in case (i) and $f(n) = O(n^\alpha)$ in case (ii), so the Dirichlet series $F(s)$ has finite abscissa of convergence, and Theorem 4.10 can therefore be applied in both cases.

(i) If $M(f,x) = O(x^\theta)$, then the integrand in the integral in (4.9) is of order $O(x^{\theta-\sigma-1})$. Hence, for any $\epsilon > 0$, this integral is uniformly convergent in $\sigma \ge \theta+\epsilon$, so it represents a function that is analytic in $\sigma \ge \theta+\epsilon$ for every $\epsilon > 0$, and thus analytic in the half-plane $\sigma > \theta$. Consequently, the function on the right of (4.9), and therefore $F(s)$, is analytic in this half-plane as well.

(ii) If $M(f,x) = Ax^\alpha + O(x^\theta)$, we set $M(f,x) = Ax^\alpha + M_1(f,x)$, and split the integral on the right of (4.9) into a sum of two integrals corresponding to the terms Ax^α and $M_1(f,x)$. Since $M_1(f,x) = O(x^\theta)$, the second of these integrals is analytic in $\sigma > \theta$ by the above argument. The first integral is

$$\int_1^\infty Ax^{\alpha-s-1}dx = \frac{A}{s-\alpha}.$$

Thus,

$$F(s) = \frac{As}{s-\alpha} + F_1(s),$$

where $F_1(s)$ is analytic in $\sigma > \theta$, as claimed. □

Theorem 4.13 can be used, in conjunction with known analytic properties of the zeta function, to obtain *lower* bounds on error terms in the various (equivalent) versions of the PNT.

We illustrate this in the case of the summatory function of the Moebius function, $M(\mu,x) = \sum_{n\le x}\mu(n)$. The PNT is equivalent to the estimate

$M(\mu, x) = o(x)$, but since $\mu(n)$ takes on the values $0, \pm 1$ in a seemingly random manner, one might expect that the "true" order of $M(\mu, x)$ is much smaller. Indeed, if the values ± 1 on squarefree integers were assigned in a truly random manner, the rate of growth of the summatory function $M(\mu, x)$ would be roughly $\sqrt{x}$, with probability close to 1.

To investigate the consequences of such estimates, suppose that, for some $\theta \geq 0$, we have

$$M(\mu, x) = O_\theta(x^\theta). \tag{4.13}$$

Theorem 4.13 then implies that the Dirichlet series for the Moebius function, namely $\sum_{n=1}^\infty \mu(n) n^{-s} = 1/\zeta(s)$, is analytic in the half-plane $\sigma > \theta$. This in turn implies that $\zeta(s)$ is meromorphic in this half-plane and satisfies

$$\zeta(s) \text{ has no zeros in } \sigma > \theta. \tag{4.14}$$

Thus, the quality of estimates for $M(\mu, x)$, and hence the quality of the error term in the PNT, depends on the "zero-free region" of the Riemann zeta function, and specifically the values θ for which (4.14) holds. Unfortunately, very little is known in this regard. The current state of knowledge can be summarized as follows:

- (4.14) holds for $\theta > 1$. (This follows immediately from the Euler product representation for $\zeta(s)$.)
- It is known that $\zeta(s)$ has infinitely many zeros with real part 1/2, so (4.14) does not hold for any value $\theta < 1/2$. (The proof of this is not easy and beyond the scope of this course.)
- For $\theta = 1/2$, statement (4.14) is the "Riemann Hypothesis", the most famous problem in number theory. It is easily seen that (4.14) holds for $\theta = 1/2$ if and only if it holds for all $\theta > 1/2$.
- It is not known whether there exists *some* θ with $1/2 \leq \theta < 1$ for which (4.14) holds (which would be a weak form of the Riemann Hypothesis).

From these remarks and Theorem 4.13 we have the following result.

Theorem 4.14 (Lower bounds for the error term in the PNT)**.** The estimate (4.13) does not hold for any $\theta < 1/2$. If it holds for $\theta = 1/2 + \epsilon$, for any $\epsilon > 0$, then the Riemann Hypothesis follows.

Remarks. (i) By a similar argument, using part (ii) of Theorem 4.13 and the fact that the Dirichlet series for $\Lambda(n)$ is $-\zeta'(s)/\zeta(s)$, one can relate the error term in the estimate $M(\Lambda, x) = x + o(x)$ to the zerofree region (4.14) and show that an estimate of the form

$$M(\Lambda, x) = x + O_\theta(x^\theta) \tag{4.15}$$

implies (4.14). Since (4.14) known to be false when $\theta < 1/2$, (4.15) does not hold if $\theta < 1/2$. Moreover, if (4.15) holds for $\theta = 1/2 + \epsilon$, for every $\epsilon > 0$, then the Riemann Hypothesis follows.

(ii) The converse of the above statements also holds, though this requires an entirely different argument, which is beyond the scope of this course. Namely, if the Riemann Hypothesis is true, then (4.13) and (4.15) hold for any $\theta > 1/2$, i.e., the error terms in these estimates are essentially (namely, up to a factor x^ϵ) of size $O(\sqrt{x})$. Since such a "squareroot bound" is characteristic of a random sequence of values ± 1, the Riemann Hypothesis can thus be interpreted as saying that the values of the Moebius function behave (essentially) "randomly".

4.4.4 Evaluation of Mertens' constant

As a final application of Theorem 4.10, we now evaluate the constant in Mertens' formula (Theorem 3.4), which we had proved in Chapter 3 in the form

$$\prod_{p\le x}\left(1-\frac{1}{p}\right) = \frac{e^{-C}}{\log x}\left(1+O\left(\frac{1}{\log x}\right)\right) \quad (x \ge 2), \tag{4.16}$$

with an unspecified constant C. We will now show this constant is equal to the Euler constant γ, as claimed in Theorem 3.4.

Taking logarithms in (4.16), we see that (4.16) is equivalent to

$$-\log P(x) = \log\log x + C + \left(\frac{1}{\log x}\right) \quad (x \ge 2), \tag{4.17}$$

where

$$P(x) = \prod_{p\le x}\left(1-\frac{1}{p}\right).$$

Now,

$$\begin{aligned}
-\log P(x) &= -\sum_{p\le x}\log\left(1-\frac{1}{p}\right)\\
&= \sum_{p\le x}\sum_{m=1}^{\infty}\frac{1}{mp^m}\\
&= \sum_{p^{\le} x}\frac{1}{mp^m} + O\left(\sum_{p\le x}\sum_{m>\log x/\log p}\frac{1}{p^m}\right)\\
&= \sum_{p^{\le} x}\frac{1}{mp^m} + O\left(\sum_{p\le x}\frac{1}{x}\right)\\
&= L(f,x) + O\left(\frac{\pi(x)}{x}\right)\\
&= L(f,x) + O\left(\frac{1}{\log x}\right),
\end{aligned}$$

where f is the function defined by

$$f(n) = \begin{cases} \frac{1}{m} & \text{if } n = p^m,\\ 0 & \text{otherwise,}\end{cases}$$

and $L(f,x) = \sum_{n\le x} f(n)/n$ is the "logarithmic" summatory function of f, as defined in Theorem 4.10. Thus, (4.17) is equivalent to

$$L(f,x) = \log\log x + C + \left(\frac{1}{\log x}\right) \quad (x \ge 2). \tag{4.18}$$

We now relate f to the Riemann zeta function. Let s be real and greater than 1. Expanding $\zeta(s)$ into an Euler product and taking logarithms, we obtain

$$\begin{aligned}
\log\zeta(s) &= \log\prod_p\left(1-\frac{1}{p^s}\right)^{-1} = \sum_p \log\left(1-\frac{1}{p^s}\right)^{-1}\\
&= \sum_p\sum_{m=1}^{\infty}\frac{1}{mp^{ms}} = F(s),
\end{aligned}$$

where $F(s)$ is the Dirichlet series of $f(n)$. On the other hand, Corollary 4.12

gives

$$\begin{aligned}\log\zeta(s) &= \log\left(\frac{1}{s-1}\left(1+O(s-1)\right)\right)\\ &= \log\frac{1}{s-1} + \log(1+O(s-1))\\ &= \log\frac{1}{s-1} + O(s-1) \quad (1<s<s_0),\end{aligned}$$

for a suitable s_0 with $1 < s_0 < 3/2$. Thus, we have

$$F(s) = \log\frac{1}{s-1} + O(s-1) \quad (1<s<s_0). \tag{4.19}$$

Next, we apply Theorem 4.10 to express $F(s)$ as an integral over $L(f,x)$. Since $0 \le f(n) \le 1$, the abscissa of convergence of $F(s)$ is ≤ 1, so the representation given by this theorem is valid in $\sigma > 1$. Noting that $L(f,x) = 0$ for $x < 2$, we obtain, in the half-plane $\sigma > 1$,

$$F(s) = (s-1)\int_2^\infty L(f,x)x^{-s}dx.$$

Substituting the estimate (4.18) for $L(f,x)$, we get

$$\begin{aligned}F(s) &= (s-1)\int_2^\infty \left(\log\log x + C + O\left(\frac{1}{\log x}\right)\right)x^{-s}dx\\ &= (s-1)\int_{\log 2}^\infty \left(\log u + C + O\left(\frac{1}{u}\right)\right)e^{-u(s-1)}du\\ &= \int_{(s-1)\log 2}^\infty \left(\log\frac{1}{s-1} + \log v + C + O\left(\frac{s-1}{v}\right)\right)e^{-v}dv.\end{aligned} \tag{4.20}$$

We now restrict s to the interval $1 < s < s_0 (< 3/2)$ and estimate the integral on the right of (4.20). The contribution of the O-term to this integral is bounded by

$$\begin{aligned}&\ll (s-1)\int_{(s-1)\log 2}^\infty \frac{e^{-v}}{v}dv\\ &\le (s-1)\left(\log\frac{1}{(s-1)\log 2} + \int_1^\infty e^{-v}dv\right)\\ &\le (s-1)\left(\log\frac{1}{s-1} + O(1)\right) \ll (s-1)\log\frac{1}{s-1},\end{aligned}$$

since $1 \ll \log 1/(s-1)$ by our assumption $1 < s < s_0 < 3/2$. In the integral over the terms $\log 1/(s-1) + \log v + C$ we replace the lower integration limit by 0, which introduces an error of order

$$\ll \int_0^{(s-1)\log 2} \left(\log \frac{1}{s-1} + |\log v| + |C| \right) dv \ll (s-1) \log \frac{1}{s-1}.$$

With these estimates, (4.20) becomes

$$\begin{aligned} F(s) &= \log \frac{1}{s-1} \int_0^\infty e^{-v} dv + \int_0^\infty (\log v) e^{-v} dv + C \int_0^\infty e^{-v} dv \\ &\quad + O\left((s-1) \log \frac{1}{s-1} \right) \\ &= \log \frac{1}{s-1} + I + C + O\left((s-1) \log \frac{1}{s-1} \right), \end{aligned} \tag{4.21}$$

where

$$I = \int_0^\infty (\log v) e^{-v} dx$$

Equating the estimates (4.21) and (4.19) for $F(s)$ we get

$$\log \frac{1}{s-1} + O(s-1) = \log \frac{1}{s-1} + I + C + O\left((s-1) \log \frac{1}{s-1} \right) \quad (1 < s < s_0).$$

Letting $s \to 1+$, the error terms here tends to zero, and we therefore conclude that

$$C = -I = -\int_0^\infty (\log v) e^{-v} dv.$$

The integral I can be found in many standard tables of integrals (e.g., Gradsheyn and Ryzhik, "Table of integrals, series, and products") and is equal to $-\gamma$. Hence $C = \gamma$, which is what we wanted to show.

4.5 Inversion formulas

In this section, we consider the converse problem of representing the partial sums $M(f, x)$ in terms of the Dirichlet series $F(s)$. This is a more difficult problem than that of expressing $F(s)$ in terms of $M(f, x)$, and the resulting formulas are more complicated, involving complex integrals, usually in truncated form with error terms, because of convergence problems. However, such "inversion formulas" are essential in applications such as the analytic proof of the prime number theorem, since in those applications analytic information on the generating Dirichlet series of an arithmetic function is available and one needs to translate that information into information on the behavior of the partial sums of the arithmetic function.

Formulas expressing $M(f,x)$, or similar functions, in terms of $F(s)$, are collectively known as "Perron formulas". We prove here two such formulas, one for $M(f,x)$, and the other for an average version of $M(f,x)$, defined by

$$M_1(f,x) = \int_1^x M(f,y)dy = \sum_{n\le x} f(n)(x-n). \tag{4.22}$$

(The second identity here follows by writing $M(f,y) = \sum_{n\le y} f(n)$ and inverting the order of summation and integration.)

The proof of these formulas rests on the evaluation of certain complex integrals, which we state in the following lemma.

Lemma 4.15. *Let $c>0$, and for $T>0$ and $y>0$ set*

$$I(y,T) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{y^s}{s} ds, \quad I_1(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{y^s}{s(s+1)} ds. \tag{4.23}$$

(i) Given $T>0$ and $y>0$, $y\ne 1$, we have

$$\begin{cases} |I(y,T)-1| \le \dfrac{y^c}{\pi T\log y} & \text{if } y>1,\\ |I(y,T)| \le \dfrac{y^c}{\pi T|\log y|} & \text{if } 0<y<1. \end{cases} \tag{4.24}$$

(ii) For all $y>0$ we have

$$I_1(y) = \begin{cases} \left(1-\frac{1}{y}\right) & \text{if } y>1,\\ 0 & \text{if } 0<y\le 1. \end{cases} \tag{4.25}$$

Proof. (i) Suppose first that $y>1$; we seek to estimate $|I(y,T)-1|$. Given $b<0$, we apply the residue theorem, replacing the path $[c-iT,c+iT]$ by the path consisting of the two horizontal segments $[c-iT,b-iT]$ and $[b+iT,c+iT]$ and the vertical segment $[b-iT,b+iT]$. In doing so, we pick up a residue equal to 1 from the pole of the integrand y^s/s at $s=0$. It remains to estimate the integral over the new path. On the vertical segment $[b-iT,b+iT]$, the integrand is bounded by $\le |y^s|/|s| \le y^b/|b|$, and so the integral over $[b-iT,b+iT]$ is bounded by $\le 2Ty^b/|b|$. Since $y>1$, this bound tends to 0 as $b\to-\infty$.

On the two horizontal segments we have $|y^s/s| \le y^\sigma/T$, so the integral over each of these two segments is bounded by

$$\le \frac{1}{2\pi}\int_b^c \frac{y^\sigma}{T}d\sigma \le \frac{1}{2\pi}\int_{-\infty}^c \frac{y^\sigma}{T}d\sigma = \frac{y^c}{2\pi T\log y}.$$

Letting $b \to -\infty$, we conclude that, for $y > 1$, the integral $I(y,T)$ differs from 1 by at most twice the above bound, i.e., an amount $\le (1/\pi)y^c/(T\log y)$, as claimed.

In the case $0 < y < 1$, we apply a similar argument, except that we now move the path of integration to a line $\sigma = a$ to the right of the line $\sigma = c$, with the new path consisting of the horizontal segments $[c - iT, a - iT]$ and $[a + iT, c + iT]$ and the vertical segment $[a - iT, a + iT]$. As before, the contribution of the vertical segment tends to 0 on letting $a \to \infty$, whereas the contribution of each of the horizontal segments is at most $\le (1/2\pi)\int_c^\infty (y^\sigma/T)d\sigma \le y^c/(2\pi T|\log y|)$. This time, however, there is no residue contribution, since the integrand has no poles in the region enclosed by the old and new paths of integration. Hence, for $0 < y < 1$, we have $|I(y,T)| \le y^c/(\pi T\log|y|)$.

(ii) Considering first the integral over a finite line segment $[c-iT, c+iT]$ and treating this integral as that of (i) by shifting the path of integration, we obtain in the case $y > 1$ a contribution coming from the residues of $y^s/(s(s+1))$ at the poles $s = 0$ and $s = -1$, namely $1 - 1/y$, and an error term that tends to 0 as $T \to \infty$. Letting $T \to \infty$, we conclude that $I_1(y) = (1-1/y)$ in the case $y > 1$. If $0 < y < 1$, the same argument applies, but without a residue contribution, so in this case we have $I_1(y) = 0$. Hence (4.25) holds for all $y > 0$ except possibly $y = 1$. To deal with the remaining case $y = 1$, we use a continuity argument: It is easily verified that the left and right-hand sides of (4.25) are continuous functions of $y > 0$. Since both sides are equal for $y > 1$, if follows that the equality persists when $y = 1$. □

We are now ready to state the two main results of this section, which give formulas for $M_1(f,x)$ and $M(f,x)$ as complex integrals over $F(s)$.

Theorem 4.16 (Perron Formula for $M_1(f,x)$)**.** *Let $f(n)$ be an arithmetic function, and suppose that the Dirichlet series $F(s) = \sum_{n=1}^\infty f(n)n^{-s}$ has finite abscissa of absolute convergence σ_a. Let $M_1(f,x)$ be defined by (4.22) and (4.7). Then we have, for any $c > \max(0,\sigma_a)$ and any real number $x \ge 1$,*

$$M_1(f,x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\frac{x^{s+1}}{s(s+1)}ds. \tag{4.26}$$

Remark. Since, on the line of integration, $|x^{s+1}| = x^{c+1}$ and $|F(s)| \le \sum_{n=1}^\infty |f(n)|n^{-c} < \infty$, the integrand is bounded by $\ll x^{c+1}/|s|^2$. Thus, the integral in (4.26) converges absolutely. By contrast, the formula for $M(f,x)$ (see Theorem 4.17 below) involves an integral over $F(s)x^{s+1}/s$, which is only conditionally convergent.

Proof. Ignoring questions of convergence for the moment, we obtain (4.26) by writing the right-hand side as

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\sum_{n=1}^\infty \frac{f(n)}{n^s}\frac{x^{s+1}}{s(s+1)}ds &= \frac{1}{2\pi i}\sum_{n=1}^\infty f(n)\int_{c-i\infty}^{c+i\infty}\frac{(x/n)^s x}{s(s+1)}ds \\ &= \sum_{n=1}^\infty f(n)xI_1(x/n) = \sum_{n\le x} f(n)(x-n) = M_1(f,x),\end{aligned}$$

using the evaluation of $I_1(y)$ given by Lemma 4.15. To justify the interchanging of the order of integration and summation, we note that

$$\begin{aligned}&\int_{c-i\infty}^{c+i\infty}\sum_{n=1}^\infty \frac{|f(n)|}{|n^s|}\left|\frac{x^{s+1}}{s(s+1)}\right|\cdot |ds| \\ &\qquad\le x^{c+1}\sum_{n=1}^\infty \frac{|f(n)|}{n^c}\int_{c-i\infty}^{c+i\infty}\frac{x^{c+1}}{|s(s+1)|}|ds| < \infty,\end{aligned}$$

since, by the assumption $c > \max(0,\sigma_a)$, we have $\sum_{n=1}^\infty |f(n)|n^{-c} < \infty$ and

$$\int_{c-i\infty}^{c+i\infty}\frac{1}{|s(s+1)|}|ds| \le \int_{c-i\infty}^{c+i\infty}\frac{1}{|s|^2}|ds| = \int_{-\infty}^{\infty}\frac{1}{c^2+t^2}dt < \infty. \qquad \square$$

Theorem 4.17 (Perron Formula for $M(f,x)$)**.** *Let $f(n)$ be an arithmetic function, and suppose that the Dirichlet series $F(s) = \sum_{n=1}^\infty f(n)n^{-s}$ has finite abscissa of absolute convergence σ_a. Then we have, for any $c > \max(0,\sigma_a)$ and any non-integral value $x > 1$,*

$$M(f,x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\frac{x^s}{s}ds, \tag{4.27}$$

where the improper integral $\int_{c-i\infty}^{c+i\infty}$ is to be interpreted as the symmetric limit $\lim_{T\to\infty}\int_{c-iT}^{c+iT}$. Moreover, given $T > 0$, we have

$$M(f,x) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} F(s)\frac{x^s}{s}ds + R(T), \tag{4.28}$$

where

$$|R(T)| \le \frac{x^c}{T} \sum_{n=1}^{\infty} \frac{|f(n)|}{n^c |\log(x/n)|}. \tag{4.29}$$

Remark. The restriction to non-integral values of x in the "infinite" version of Perron's formula (4.27) can be dropped if we replace the function $M(f,x)$ by the interpolation between its left and right limits, namely $M^*(f,x) = (1/2)(M(f,x-)+M(f,x+))$. This can be proved in the same manner using the following evaluation of the integral $I(y,T)$ in the case $y=1$:

$$\begin{aligned} I(1,T) &= \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{1}{s} ds = \frac{1}{2\pi}\int_{-T}^{T} \frac{c-it}{c^2+t^2} dt \\ &= \frac{1}{2\pi}\int_{-T}^{T} \frac{c}{c^2+t^2} dt = \frac{1}{2\pi}\int_{-T/c}^{T/c} \frac{1}{1+u^2} du \\ &= \frac{1}{2\pi}\left(\arctan(T/c) - \arctan(-T/c)\right), \end{aligned}$$

which converges to $(1/2\pi)(\pi/2 - (-\pi/2)) = 1/2$ as $T \to \infty$.

However, in applications the stated version is sufficient, since for any integer N, $M(f,N)$ is equal to $M(f,x)$ for $N < x < N+1$ and one can therefore apply the formula with such a non-integral value of x. Usually one takes x to be of the form $x = N + 1/2$ in order to minimize the effect a small denominator $\log(x/n)$ on the right-hand side of (4.29) can have on the estimate.

Proof. The formula (4.27) follows on letting $T \to \infty$ in (4.28), so it suffices to prove the latter formula. To this end we proceed as in the proof of Theorem 4.16, substituting $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ in the first (main) term on the right-hand side of (4.28), to obtain

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} F(s)\frac{x^s}{s} ds = \sum_{n=1}^{\infty} f(n) I(x/n, T).$$

The interchanging of integration and summation is again permissible, since the range of integration is a compact interval, $[c-iT, c+iT]$, and the series $\sum_{n=1}^{\infty} f(n)n^{-s}$ converges absolutely and uniformly on that interval. Estimating $I(x/n,T)$ by Lemma 4.15, we obtain

$$\sum_{n=1}^{\infty} f(n) I(x/n,T) = \sum_{n\le x} f(n) + E(T) = M(f,x) + E(T),$$

where

$$|E(T)| \leq \sum_{n=1}^{\infty} |f(n)| \frac{(x/n)^c}{T|\log(x/n)|} = \frac{x^c}{T} \sum_{n=1}^{\infty} \frac{|f(n)|}{n^{\sigma}|\log(x/n)|}.$$

Collecting these estimates yields (4.28), with $R(T) = -E(T)$ satisfying (4.29), as required. □

4.6 Exercises

4.1 Let $F(s) = \sum_{m,n=1}^{\infty} [m,n]^{-s}$. Determine the abscissa of convergence σ_c of $F(s)$ and express $F(s)$ in terms of the Riemann zeta function. (Hint: Express $F(s)$ as $\sum_{n=1}^{\infty} f(n)n^{-s}$, where $f(n) = \#\{(a,b) \in \mathbb{N}^2 : [a,b] = n\}$, and represent the latter as an Euler product.)

4.2 Express the Dirichlet series $\sum_{n=1}^{\infty} d(n)^2 n^{-s}$ in terms of the Riemann zeta function. Then use this relation to derive a convolution identity relating the functions $d^2(n)$ and $d_4(n)$ (where $d_k(n)\#\{(a_1,\dots,a_k) \in \mathbb{N}^k : a_1 \dots a_k = n\}$ is the generalized divisor function).

4.3 Let $f(n) = \sum_{d|n} (\log d)/d$, and let $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$. Evaluate $F(s)$ in terms of the Riemann zeta function.

4.4 Evaluate the series $\sum_{(m_1,\dots,m_r)=1} m_1^{-s} \cdots m_r^{-s}$, where the summation is over all tuples $(m_1,\dots,m_r)$ of positive integers that are relatively prime, in terms of the Riemann zeta function.

4.5 Let $f(n)$ be the unique positive real-valued arithmetic function that satisfies $\sum_{d|n} f(d)f(n/d) = 1$ for all n (i.e., f is the positive "Dirichlet square root" of the function 1). Let $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ be the Dirichlet series of $F(s)$.

   (i) Express $F(s)$ for $\sigma > 1$ in terms of the Riemann zeta function.
   (ii) Find an explicit formula for $f(p^k)$, where p is prime and $k \geq 1$.

4.6 For each of the following functions $f(n)$ determine the abscissa of convergence σ_c and the abscissa of absolute convergence σ_a of the associated Dirichlet series.

   (i) $f(n) = \omega(n)$ (where $\omega(n)$ is the number of distinct prime factors of n)
   (ii) $f(n) = e^{2\pi i \alpha n}$, where $\alpha \in \mathbb{R} \setminus \mathbb{Z}$
   (iii) $f(n) = n^{i\alpha n}$, where $\alpha \in \mathbb{Z}$
   (iv) $f(n) = d_k(n) = \#\{(a_1,\dots,a_k) \in \mathbb{N}^k : a_1 \dots a_k = n\}$ (the generalized divisor function)
   (v) $f(n)$ any periodic function with period q and $\sum_{n=1}^{q} f(n) = 0$.

4.7 Let σ_1 and σ_2 be real numbers with $\sigma_1 \le \sigma_2 \le \sigma_1 + 1$. Construct an arithmetic function whose Dirichlet series has abscissa of convergence $\sigma_c = \sigma_1$ and abscissa of absolute convergence $\sigma_a = \sigma_2$.

4.8 Let $f(n)$ be an arithmetic function satisfying

$$S(x) = \sum_{n \le x} f(n) = Ax^{\alpha} + Bx^{\beta} + O(x^{\delta}) \quad (x \ge 1),$$

where $\alpha > \beta > \delta \ge 0$ are real numbers and A and B are *non-zero* real numbers. Let $F(s) = \sum_{n \ge 1} f(n)n^{-s}$ be the generating Dirichlet series for f. Find, with proof, a half-plane (as large as possible) in which $F(s)$ is guaranteed to have a meromorphic continuation, and determine all poles (if any) of (the meromorphic continuation of) $F(s)$ in that region, and the residues of F at those poles.