6 Continued fractions

6.1 Definitions and notations

Definition 6.1 (Continued fractions). *A finite or infinite expression of the form*

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}, \tag{6.1}$$

*where the a_i are real numbers, with $a_1, a_2, \dots > 0$, is called a **continued fraction** (c.f.). The numbers a_i are called the **partial quotients** of the c.f.*

*The continued fraction (6.1) is called **simple** if the partial quotients a_i are all integers. It is called **finite** if it terminates, i.e., if it is of the form*

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots + \cfrac{1}{a_n}}}, \tag{6.2}$$

*and **infinite** otherwise.*

Notation (Bracket notation for continued fractions). *The continued fractions (6.1) and (6.2) are denoted by $[a_0, a_1, a_2, \dots]$ and $[a_0, a_1, a_2, \dots, a_n]$, respectively. In particular,*

$$[a_0] = a_0, \quad [a_0, a_1] = a_0 + \frac{1}{a_1}, \quad [a_0, a_1, a_2] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}}, \quad \dots$$

Remarks. (i) Note that the first term, a_0, is allowed to be negative or 0, but all subsequent terms a_i must be positive. This requirement ensures that there are no zero denominators and that any finite c.f. (6.2), and all of its convergents, are well-defined.

(ii) In the sequel we will focus on the case of simple c.f.'s, i.e., c.f.'s where all partial quotients are integers.

6.2 Convergence of infinite continued fractions

Definition 6.2 (Convergents). *The **convergents** of a (finite or infinite) c.f. $[a_0, a_1, a_2, \dots]$ are defined as*

$$C_0 = [a_0], \ C_1 = [a_0, a_1], \ C_2 = [a_0, a_1, a_2], \dots$$

If the c.f. is simple, its convergents C_i represent rational numbers, denoted by

$$C_i = \frac{p_i}{q_i},$$

where p_i/q_i is in reduced form.

Definition 6.3 (Convergence of infinite continued fractions). *An infinite c.f. $[a_0, a_1, a_2, \dots]$ is called **convergent** if its sequence of convergents $C_i = [a_0, a_1, \dots, a_i]$ converges in the usual sense, i.e., if the limit*

$$\alpha = \lim_{i \to \infty} C_i = \lim_{i \to \infty} [a_0, a_1, \dots, a_i]$$

exists (and is a real number). In this case, we say that the continued fraction $[a_0, a_1, a_2, \dots]$ ***represents*** *the number* α*, or is a* ***continued fraction expansion*** *of* α*, and we write*

$$\alpha = [a_0, a_1, a_2, \dots].$$

Theorem 6.4 (Convergence of infinite simple c.f.'s)**.** *Any infinite simple c.f.* $[a_0, a_1, \dots]$ *is convergent and thus represents some real number.*

6.3 Properties of Convergents

Proposition 6.5 (Formulas for p_i and q_i)**.** *Let* $\alpha = [a_0, a_1, \dots]$ *be a simple c.f. with convergents* $C_i = [a_0, a_1, \dots, a_i] = \frac{p_i}{q_i}$.

(i) ***Recursion formula:*** *The numbers* p_i *and* q_i *are given by the recurrence*

$$p_i = a_i p_{i-1} + p_{i-2},$$
$$q_i = a_i q_{i-1} + q_{i-2}$$

for $i = 1, 2, \dots$*, along with the initial conditions* $p_0 = a_0, p_{-1} = 1, q_0 = 1, q_{-1} = 0$.

(ii) ***Matrix representation:*** *For* $i = 0, 1, 2, \dots$

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_i & p_{i-1} \\ q_i & q_{i-1} \end{pmatrix}$$

Theorem 6.6 (Properties of convergents)**.** *The convergents* $C_i = p_i/q_i$ *of an infinite simple continued fraction* $\alpha = [a_0, a_1, a_2, \dots]$ *satisfy:*

(i) $(p_i, q_i) = 1$ *for* $i = 0, 1, \dots$*; i.e., the fractions* p_i/q_i *are reduced.*

(ii) $q_1 < q_2 < \cdots$*; i.e., for* $i \geq 1$*, the denominators* q_i *are strictly increasing.*

(iii) $C_0 < C_2 < C_4 < \cdots < \alpha < \cdots < C_5 < C_3 < C_1$. *That is, the even-indexed convergents form an increasing sequence, while the odd-indexed convergents form a decreasing sequence, with the value of the c.f. sandwiched between both sequences.*

(iv) $C_{i+1} - C_i = \dfrac{(-1)^i}{q_i q_{i+1}}$ *for* $i = 0, 1, 2, \dots$.

(v) $\left| \dfrac{p_i}{q_i} - \alpha \right| < \dfrac{1}{q_i q_{i+1}}$ *for* $i = 0, 1, 2, \dots$.

(vi) ***Best approximation property:*** *For any rational number* a/b *with* $a \in \mathbf{Z}$, $b \in \mathbf{N}$*, and* $1 \leq b \leq q_i$,

$$\left| \frac{p_i}{q_i} - \alpha \right| \leq \left| \frac{a}{b} - \alpha \right|,$$

with equality if and only if $a/b = p_i/q_i$. *That is, the convergent* p_i/q_i *is the best-possible approximation to* α *among all rational numbers with the same or smaller denominator.*

6.4 Expansions of real numbers into continued fractions

Proposition 6.7 (Continued fraction algorithm)**.** *Given a real number α, define successively real numbers $\alpha_0, \alpha_1, \dots$, and integers $a_0, a_1, \dots$ by*

$$\begin{aligned} \alpha_0 &= \alpha, & a_0 &= [\alpha_0], \\ \alpha_1 &= \frac{1}{\alpha_0 - [\alpha_0]}, & a_1 &= [\alpha_1], \\ \alpha_2 &= \frac{1}{\alpha_1 - [\alpha_1]}, & a_2 &= [\alpha_2], \\ &\dots & &\dots \end{aligned}$$

where $[x]$ denotes the integer part of x (i.e., the "floor function"). Stop the algorithm if α_n is an integer (and thus $a_n = \alpha_n$); otherwise continue indefinitely. Then $[a_0, a_1, \dots]$ is a simple c.f. that represents the number α. Moreover, for any $i \geq 0$ we have

$$\alpha_i = [a_i, a_{i+1}, \dots], \quad \alpha = [a_0, a_1, \dots a_{i-1}, \alpha_i].$$

Theorem 6.8 (Continued fraction expansion of rational numbers)**.** *Any finite simple c.f. represents a rational number. Conversely, any rational number α can be expressed as a simple finite c.f. $\alpha = [a_0, a_1, \dots, a_n]$. Moreover, under the requirement that $a_n > 1$, this representation is unique. Thus, there is a one-to-one correspondence between rational numbers and finite simple c.f.'s with last partial quotient greater than 1.*

Theorem 6.9 (Continued fraction expansion of irrational numbers)**.** *Any infinite simple c.f. represents an irrational number. Conversely, any irrational number α can be expressed as a simple* infinite *c.f. $\alpha = [a_0, a_1, a_2, \dots]$, and this representation is unique. Thus, there is a one-to-one correspondence between irrational numbers and infinite simple c.f.'s.*

Theorem 6.10 (Continued fraction expansion of quadratic irrationals)**.** *The c.f. expansion of a quadratic irrational (i.e., a solution of a quadratic equation with integer coefficients) is eventually periodic, i.e., of the form*

$$[a_0, \dots, a_N, \overline{a_{N+1}, \dots, a_{N+p}}],$$

where the bar indicates the periodic part. Conversely, any infinite simple c.f. that is eventually periodic represents a quadratic irrational. Thus, there is a one-to-one correspondence between quadratic irrationals and infinite, eventually periodic simple c.f.'s.