Cardinality: Countable and Uncountable Sets

- **Tool: Bijections.**
  - **Bijection from a set S:** Let A and B be sets. A bijection from A to B is a function \( f : A \to B \) that is both injective and surjective.
  - **Some properties of bijections:**
    * **Inverse functions:** The inverse function of a bijection is a bijection.
    * **Compositions:** The composition of bijections is a bijection.

- **Definitions.**
  - **Cardinality:** Two sets A and B are said to have the same cardinality if there exists a bijection from A to B.
  - **Finite and infinite sets:** A set is called finite if it is empty or has the same cardinality as the set \( \{1, 2, \ldots, n\} \) for some \( n \in \mathbb{N} \); it is called infinite otherwise.
  - **Countable sets:** A set is called countable (or countably infinite) if it has the same cardinality as \( \mathbb{N} \). Equivalently, a set A is countable if it can be enumerated in a sequence, i.e., if all of its elements can be listed as a sequence \( a_1, a_2, \ldots \). A set is called uncountable if it is infinite and not countable.

- **Two famous results with memorable proofs.**

  - **The rational numbers are countable.**
    
    **Proof idea:** “Zigzag method”. Arrange the rationals in matrix and enumerate by traversing the matrix in zigzag fashion. (See text, p. 90).

  - **The real numbers are uncountable.**
    
    **Proof idea:** “Cantor’s diagonalization method”. Assuming the reals are countable, the decimal expansions of all real numbers in \([0, 1)\) can be put in a matrix with countably many rows and columns. Use the digits in the diagonal to construct a real number in \([0, 1)\) not accounted for, thus obtaining a contradiction. (See text, p. 266.)

- **Some general results on countable and uncountable sets.**
  - **An infinite subset of a countable set is countable; a superset of an uncountable set is uncountable.**
    **Proof idea:** For the first statement, assume A is a countable set and \( B \subseteq A \) an infinite subset of A. Enumerate A as a sequence \( a_1, a_2, \ldots \). The subsequence consisting of those elements that belong to B then is an enumeration of B. The second statement follows by contraposition. (A “superset” is the opposite of the subset relation, i.e., \( B \) is a superset of \( A \) if \( A \subseteq B \)).

  - **A finite or countable union of countable sets is countable.** In other words, if \( A_1, A_2, \ldots, A_k \) are each countable, then so is \( A_1 \cup A_2 \cup \ldots A_k \), and the same holds for unions of countably many countable sets: If \( A_1, A_2, A_3, \ldots \) are each countable, then so is the union \( A_1 \cup A_2 \cup A_3 \cup \ldots \).
    **Proof idea:** For two countable sets A and B, enumerate the sets as \( A = \{a_1, a_2, a_3, \ldots\} \) and \( B = \{b_1, b_2, b_3, \ldots\} \), and “interlace” these enumerations to get an enumeration of the union: \( A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, \ldots\} \). The same method works for unions of finitely many countable sets \( A_1, \ldots, A_k \).
    For unions of countably many sets \( A_1, A_2, A_3, \ldots \), use the “zigzag” trick: Enumerate each \( A_i \) as \( \{a_{i1}, a_{i2}, \ldots\} \), arrange these elements in an infinite matrix and traverse this matrix in “zigzag” fashion to get an enumeration of all elements \( a_{ij}, i \in \mathbb{N}, j \in \mathbb{N} \).

  - **The cartesian product of finitely many countable sets is countable.** In other words, if \( A_1, \ldots, A_k \) are countable, then so is \( A_1 \times \cdots \times A_k \).
    **Proof idea:** For the case of two countable sets A and B, enumerate these sets as \( A = \{a_1, a_2, \ldots\} \) and \( B = \{b_1, b_2, \ldots\} \), arrange the elements \( (a_i, b_j) \) of \( A \times B \) in an infinite matrix and use a “zigzag” argument to traverse this matrix and obtain an enumeration of all matrix elements. To extend this result to the product of \( k \) countable sets \( A_1 \times \cdots \times A_k \), use induction on \( k \).

**Remark:** While unions of countably many countable sets are countable, cartesian products involving countably many countable (or even finite) “factors” need not be countable.

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\(^1\)Some texts (e.g., Rosen’s “Discrete Mathematics”) use the term “countable” in the sense of “finite or countably infinite”. We use here the convention of the D’Angelo/West text where the term “countable” is reserved for infinite sets.