A standard problem of representation theory is to take a given representation and break it down into simple pieces: that is, to explain how it can be built up from irreducible representations. For a simple complex Lie algebra—say, the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ of traceless $n \times n$ complex matrices with commutator—one naturally constructs a large collection of induced representations, and then one would like to know how many times each irreducible representation appears in a given induced representation.

In the 1970s, Kazhdan and Lusztig gave a conjectural description of these multiplicities and explained that the multiplicities should be intimately related to the geometry and topology of special subvarieties of flag varieties known as Schubert varieties. Around 1980, Beilinson-Bernstein and Brylinski-Kashiwara independently proved the Kazhdan-Lusztig conjecture. The method of localization introduced by Beilinson and Bernstein is especially powerful: it establishes a geometric description of the entire category of representations of the Lie algebra, by “spreading out” representations as geometric objects living on the flag variety. These geometric objects naturally have an intrinsic notion of parallel transport: they are $\mathcal{D}$-modules.

In this course, we'll develop some basics of representation theory of finite-dimensional, simple complex Lie algebras, the geometry of flag varieties, and $\mathcal{D}$-modules, and then put them all together to understand Beilinson-Bernstein localization and how it can be used to prove the Kazhdan-Lusztig conjecture.

The course will assume some cultural familiarity with complex Lie algebras and with algebraic varieties, but not with $\mathcal{D}$-modules. A student who has taken a first course in representation theory and knows what an algebraic variety is, or who has taken a course in algebraic geometry and knows a few basic definitions about Lie algebras, should be able to follow the course.