

Math 595. Hermitian forms and CR geometry

Instructor: Jiri Lebl

Spring 2010, mini course, second half of semester

This course studies applications of Hermitian forms to several complex variables and CR geometry. We will therefore touch on techniques connecting analysis in several complex variables with linear algebra and with both real and complex algebraic geometry. After introducing holomorphic functions in several variables, we show that even the most basic examples of domains in several complex variables are not biholomorphically equivalent. The geometry of the boundary of domains therefore plays an important role in the theory. Furthermore, not every domain is a natural domain of definition for a holomorphic function.

These observations lead to CR geometry, which is the study of real objects, in particular real hypersurfaces, in complex space. We focus on hypersurfaces that are real-analytic manifolds, and hence have real analytic defining functions. By introducing and using polarization we will find out that we can think of a real polynomial the complex (holomorphic) Veronese map applied to a Hermitian form. A similar more general concept applies for real analytic functions. By writing the form as a sum of rank one matrices we find that a real polynomial is a difference of squared norms of holomorphic polynomial maps. We will apply this way of thinking to answer two basic questions in several complex variables. First, we will characterize real hypersurfaces that contain complex analytic varieties of positive dimension.

Second, we will apply now classical results for Hermitian forms to the classification of degree-two rational maps between spheres in different dimensions, a recent but accessible result by the instructor. We will end the course by discussion of some combinatorial number theory and real-algebraic geometric aspects of CR mappings between spheres.

This course will be accessible to anyone with a basic knowledge of complex analysis in one variable and of abstract algebra. It should be of interest to anyone wanting to find out more about several complex variables or about applications of algebra and algebraic geometry.