1 Introduction

Suppose that \( T \) is a square tiling of a region \( D \) in \( \mathbb{C} \), i.e., \( T \) is a finite collection of squares with edges parallel to the \( x \) and \( y \) axes, that have mutually disjoint interiors and their union is all of \( \overline{D} \). A standard tiling is a tiling where every square has side-length 1 and the vertices are of the form \((m,n) \in \mathbb{Z}^2\).

\[ \text{Figure 1: Standard tiling and non-standard tilings} \]

1.1 Tiling-harmonic and graph-harmonic functions

Denote with \( \mathcal{F}(T) \) the set of all real valued functions defined on the set of vertices of \( T \). For such a function \( u \), define the energy of \( u \) on \( T \) to be the non-negative number

\[ E_T(u) = \sum_{t \in T} (\max_p u(p) - \min_p u(p))^2, \]

where \( t \) is a square in \( T \) and \( p \) runs over all vertices of \( T \) that lie on \( t \). A function \( u \in \mathcal{F}(T) \) is called \( T \)-harmonic (or tiling-harmonic) if for every subtiling \( T' \) (i.e., a subset of \( T \)) and every function \( u' \in \mathcal{F}(T) \) such that \( u' \equiv u \) on the boundary vertices of \( T' \), we have \( E_u(T') \leq E_{u'}(T') \).

Given a function \( u \) defined on the boundary vertices of \( T \), there may be more than one tiling-harmonic extension \( U \in \mathcal{F}(T) \) of \( u \). In this example, any number in the middle vertex between 1 and 2 would give you the same minimal energy.

A function \( u \in \mathcal{F}(T) \) is called graph-harmonic if for every interior vertex \( p \) of \( T \), the value \( u(p) \) is equal to the average of \( u \) on the neighbor vertices of \( p \). Unlike tiling-harmonic functions, given a function \( u \) defined on the boundary vertices of \( T \), there is a unique graph-harmonic function \( U \in \mathcal{F}(T) \) that extends \( u \).
1.2 Project goals

We investigate the following questions.

- Suppose that $T$ is a standard square grid tiling. We want to construct an efficient computer algorithm (in a Mathematica or Java environment) such that, given a real valued function $u$ on the boundary vertices of $T$, the program computes an extension $U \in \mathcal{F}(T)$ of $u$ which minimizes the energy $E$ on $T$.

- Using the above algorithm we want to compare this notion of tiling-harmonic functions with graph-harmonic functions. More precisely, suppose that $T$ is a standard square grid tiling and $u$ is defined on the boundary vertices of $T$. Let $U, U'$ be the tiling-harmonic function, graph-harmonic function, respectively, of $T$ that extends $u$. Is $U = U'$? If not, how big is $|U(p) - U'(p)|$ when $p$ is a vertex of $T$ away from the boundary?

Similar questions can be posed for non-standard square grid tilings.

2 Algorithm for calculating energy minimizers in standard tilings

The algorithm consists of two steps. We first find energy minimizers for $2 \times 2$ tilings with 8 boundary vertices, and one interior vertex needed to be calculated. We then apply the algorithm of $2 \times 2$ tilings for $n \times m$ tilings (with $2n + 2m + 2$ boundary vertices and $nm$ interior vertices). All algorithms run in Java.

2.1 Algorithm for a $2 \times 2$ tiling

Suppose that $T$ is a $2 \times 2$ standard tiling consisting of 4 squares $S_1, \ldots, S_4$ and $u$ is defined on the boundary vertices of $T$. The goal is to find a value of $u$ in the interior vertex so that the energy is as small as possible.

Note that

$$E_T(u) = (M_1 - m_1)^2 + (M_2 - m_2)^2 + (M_3 - m_3)^2 + (M_4 - m_4)^2$$

where $M_i, m_i$ are the maximum, minimum respectively, of $u$ on $S_i$.

Let $X$ be the value of $u$ on the interior vertex that gives the smallest energy. Considering only one square $S_i$, the most ideal value of $X$ is in the interval $[m_i, M_i]$, so that $X$ does not affect the energy of $S_i$. We sort the $m_1, M_1, \ldots, m_4, M_4$ from smallest to largest and generate 7 intervals from those sorted numbers. If, for example,

$$m_1 < m_2 < M_1 < m_4 < M_2 < m_3 < M_3 < M_4,$$

then the 7 intervals will be $I_1 = [m_1, m_2], I_2 = [m_2, M_1], \ldots, I_7 = [M_3, M_4]$.

In each interval $I_k$, the algorithm generates a value $X_k$ as the candidate for the central value as follows. Suppose that $[a, b]$ is the interval $I_k$. For each square $S_i$ we compare $a, b$ with $m_i, M_i$.

- If $b \leq m_i$ then let $E_i(x) = (M_i - x)^2$, $\alpha_i = 1$ and $c_i = M_i$.
- If $m_i \leq a \leq b \leq M_i$ then let $E_i(x) = (M_i - m_i)^2$, $\alpha_i = 0$ and $c_i = 0$.
- If $M_i \leq a$ then let $E_i(x) = (x - m_i)^2$, $\alpha_i = 1$ and $c_i = m_i$. 

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If \( \alpha_1 + \cdots + \alpha_4 = 0 \) then let \( X_i = a \). If \( \alpha_1 + \cdots + \alpha_4 \neq 0 \), then define
\[
c = \frac{c_1 + c_2 + c_3 + c_4}{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4},
\]
and \( E(x) = E_1(x) + E_2(x) + E_3(x) + E_4(x) \). If \( c \) is not between \( a \) and \( b \) then \( X_k \) is the one of \( a, b \) that minimizes \( E \), or \( a \) if both have the same energy. If \( c \) is between \( a \) and \( b \) then \( X_k \) equals \( c \).

Let \( X \) be the \( X_k \), \( k = 1, \ldots, 7 \) which gives the smallest energy, or, if more than one, the smallest \( X_k \) that gives the smallest energy.

### 2.2 Algorithm for a \( n \times m \) tiling

We apply the algorithm for \( 2 \times 2 \) tilings to find energy minimizers for \( n \times m \) tilings, with \((n + 1)(m + 1)\) vertices. These vertices are denoted by \((i, j)\), with \(0 \leq i \leq n \) and \(0 \leq j \leq m\). We are given the value of \( u \) on \((0, i), (j, 0), (n, i), (m, j)\) with \(0 \leq i \leq n \) and \(0 \leq j \leq m\), and the goal is to find the values of \( u \) for \((i, j)\) with \(1 \leq i \leq n - 1 \) and \(1 \leq j \leq m - 1 \) so that the energy is minimal.

In step 1, we set up initial values for the interior vertices. The initial values can be any numbers within the maximum and the minimum of the vertices on the boundary. One can use as initial values the values of the graph harmonic functions (which can be obtained by a simple algorithm). In step 2, we apply the \( 2 \times 2 \) algorithm to replace the initial value of \((1, 1)\) with one that gives the smallest energy. Then we apply the \( 2 \times 2 \) algorithm to replace the initial value of \((2, 1)\) with one that gives the smallest energy. We repeat this procedure until we replace all \( nm \) initial values of the interior vertices with new values. In step 3, we return to point \((1, 1)\) and apply again the \( 2 \times 2 \) algorithm and we keep the same procedure. The algorithm terminates when the difference of the energy after \( N - 1 \) steps minus the energy after \( N \) steps is less than a prescribed error.

### 2.3 Examples of the algorithm

In each of the following three examples, given some boundary values, we calculate the graph-harmonic function and use our algorithm to calculate the tiling-harmonic function. Finally, we calculate the difference of the two functions. In the following figures, we present the tiling-harmonic function, graph-harmonic function and the difference, respectively.

- **Example 1**: \( u(0, j) = j, u(20, j) = j, u(0, i) = 0, u(20, i) = 20. \)

Notice that both the tiling-harmonic and the graph-harmonic lay on a plane and they are equal.

- **Example 2**: \( u(0, j) = (10 - j)^2, u(20, j) = (10 - j)^2, u(0, i) = (10 - i)^2, u(20, i) = \frac{(10 - i)^2}{(10 - i)^2}. \)
The tiling-harmonic and graph-harmonic functions create graphs with similar shapes. The differences are relatively large on the corners of the function but decrease to zero as it approaches to the center.

- **Example 3:** \( u(0, j) = j \), \( u(20, j) = 20 - j \), \( u(0, i) = i \), \( u(20, i) = 20 - i \).

The graphs of the two functions are very close. But the surface of tiling-harmonic around the diagonal lines is not as smooth as the graph-harmonic function. The tiling-harmonic function in this case is the union of four triangles, hence piecewise linear.

### 3 Conclusions and future plans

The following observations can be made after applying the algorithm on different boundary values.

- Whenever the boundary values lie on a plane \( P \), the tiling harmonic is equal to the graph harmonic function. Moreover, both lie on the plane \( P \).

- Given some boundary values, the graph-harmonic function and the tiling-harmonic function are **not in general equal**.

We hope to use our current algorithm as a stepping stone and continue to explore other features about tiling-harmonic functions. A generalized algorithm for non-standard tilings will be the next goal. Then we will be able to compare tiling-harmonic and graph-harmonic functions in broader point of view.

- Using the algorithm of standard square tilings, we want to create an algorithm for general tilings.

- We want to compare graph-harmonic functions and tiling-harmonic functions on general tilings.

- Suppose that \( T \) is a standard square grid tiling of the upper half plane. Is it true that the only non-negative \( T \)-harmonic functions \( u \) that vanish on the real vertices have the form \( u(x, y) = cy \), where \( c \geq 0 \) is a constant?