ANGULAR DISTRIBUTION OF LATTICE POINTS
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1. Spacing Statistics

We continued to investigate numerically the spacing statistics of the angular distribution of certain configurations of integer lattice points, in both Euclidean and hyperbolic geometries. The main focus was on the pair correlation function, and also on extending some of the experiments done last semester on the limiting gap distribution.

Definition 1. For every (non-ordered) list of numbers $F = (x_1, x_2, \ldots, x_n)$ in $[0,1]$ consider:

$$R_F(\xi) := \frac{1}{n} \# \left\{ 1 \leq i \neq j \leq n : 0 \leq x_i - x_j \leq \frac{\xi}{n} \right\}, \quad \xi \in [0, \infty).$$

Let $(F_n)$ be an increasing sequence of lists in $[0,1]$. The pair correlation measure of $(F_n)$ is defined by $R(\xi) := \lim_{n} R_{F_n}(\xi)$ if the limit exists, with density $g = -R'$ (called pair correlation function).

2. Farey fractions with congruence conditions

Elementary exercises in Möbius inversion shows that the sequence $(F_n)$ of Farey fractions of order $n$ is uniformly distributed. Its limiting gap distribution and pair correlation function can be expressed in closed form $[5,2]$. Both vanish on the interval $[0, \frac{2}{\pi^2}]$, as an expression of the repulsion between consecutive Farey fractions of order $n$, exhibited by the inequality $\gamma' - \gamma \geq \frac{1}{n^2}$. We investigated numerically situations where $F_n$ is replaced by subsets where congruence conditions are imposed. One common feature is the persistence of the gap, but the shape of the graphs is different, suggesting that when congruence conditions are imposed, the number of intervals where the gap density is smooth is infinite, which is consistent with the computations in $[1]$.

3. Euclidean angular distribution

For fixed (large) $Q > 0$, join $P = (x_0, y_0)$ with all points $A \in \mathbb{Z}^2 \cap \{(x, y) : x^2 + y^2 \leq Q^2\}$ in the Euclidean plane $\mathbb{R}^2$. We studied the pair correlation of the increasing family of sets $F_Q(x_0, y_0)$ of angles between the rays $PA$, of cardinality $\sim \pi Q^2$, for various positions $(x_0, y_0) \in \mathbb{Q}^2$. For generic $(x_0, y_0) \notin \mathbb{Q}^2$ it is known that the limiting gap distribution function is 1 $[4]$. In all cases the angles are uniformly distributed. Our numerical experiments show that the gap distributions depends significantly on $(x_0, y_0)$, with various repulsion strengths. It also shows a surprising feature: the spikes that appear in the pair correlation function with observer located at $(\frac{1}{n}, 0)$ appear to be periodically distributed.
4. Hyperbolic Angular Distribution

The group $G = \text{PSL}(2, \mathbb{R})$ acts on the upper half-plane $\mathbb{H}$ by $g \cdot z = \frac{az + b}{cz + d}$. The hyperbolic disk $B(i, R) = \{ z \in \mathbb{H} : \varrho(i, z) \leq R \}$ coincides with the Euclidean disk of center $2i \cosh R$ and radius $2 \sinh R \sim e^R$, with hyperbolic volume $\text{Vol} B(i, R) = \int_{B(i, R)} \frac{dx \, dy}{y^2} = 2\pi (\cosh R - 1) \sim \pi e^R$. Geodesics connecting two points are either half-circles orthogonal to the real axis, or vertical half-lines. Let $\Gamma$ be a discrete subgroup of $G$ with $\text{Vol}(\Gamma \backslash \mathbb{H}) < \infty$. Consider a fixed observer in $\mathbb{H}$ located at $\omega$. Consider the finite subset $S^\omega_R(\Gamma) = \{ \gamma \in \Gamma : \varrho(\omega, \gamma \omega) \leq R \}$ of $\Gamma$, of cardinality $\sim c_\Gamma e^R$. We investigated numerically the spacing statistics of the increasing family of (uniformly distributed) sets of angles between geodesic rays connecting $\omega$ with $\gamma \omega$, $\gamma \in S^\omega_R(\Gamma)$, when $\Gamma = \text{PSL}_2(\mathbb{Z})$. Our plots are consistent with the recent theoretical results on the pair correlation of these sets [3, 6].

References