Geodesics on Surfaces of Revolution in Minkowski Space
IGL Spring 2013 Report

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Abstract

We create visualizations of geodesics on surfaces of revolution in Minkowski space to supplement research questions related to curvature, behavior near singularities, and other interesting questions about the geometry of Lorentzian warped product spaces.

Mathematical Background

Our main objects of interest are Lorentz surfaces in Minkowski 3-space with rotational symmetry. They are obtained by taking a timelike profile curve $\beta(s) = (r(s), 0, f(s))$ with $r(s) > 0$, parameterized with respect to its arclength $s$, which is equivalent to

$$r'(s)^2 - f'(s)^2 = -1,$$  \hspace{1cm} (1)

and rotating the profile curve about the timelike axis. Alternatively, one can prescribe a radial function $r(s) > 0$ and obtain $f(s)$ by using equation 1 and integrating,

$$f(s) = \int_{s_0}^{s} \sqrt{r'(\tau)^2 + 1} \, d\tau.$$  \hspace{1cm} (2)

The parameterization of the surface is

$$F(\theta, s) = (r(s) \cos \theta, r(s) \sin \theta, f(s)).$$  \hspace{1cm} (3)

From this, one can easily compute the metric. In these coordinates, the metric is represented by the matrix

$$\begin{pmatrix}
    r(s)^2 & 0 \\
    0 & -1
\end{pmatrix}.$$

A simple exercise of computing Christoffel symbols and writing down the geodesic equations for a geodesic $\gamma$ produces

$$\ddot{\theta} + 2 \frac{r'(s)}{r(s)} \dot{\theta} \dot{s} = 0$$  \hspace{1cm} (4)

$$\ddot{s} + r'(s)r(s)\dot{\theta}^2 = 0$$  \hspace{1cm} (5)

* with graduate student leader Bill Karr and professor Stephanie Alexander.
where \( \gamma(t) = F(\theta(t), s(t)) \). In addition, we always assume without loss of generality that geodesics are unit speed. In these coordinates, this means

\[
\|\dot{\gamma}\|^2 = r(s)^2 \dot{\theta}^2 - \dot{s}^2 = -1 \quad (6)
\]

\[
\|\dot{\gamma}\|^2 = r(s)^2 \dot{\theta}^2 - \dot{s}^2 = 1 \quad (7)
\]

if \( \gamma \) is timelike or spacelike, respectively. We also consider lightlike geodesics which satisfy

\[
\|\dot{\gamma}\|^2 = r(s)^2 \dot{\theta}^2 - \dot{s}^2 = 0. \quad (8)
\]

This property is called the causal character of the geodesic \( \gamma \).

**Visualizations in MATLAB for IGL**

Our IGL project was specifically to use computers to create visualizations of geodesics on surfaces of revolution.

At the beginning of the semester, we attempted to use Mathematica to draw these surfaces with geodesics on them, but we found that MATLAB was more flexible to work with since the core of our project is solving the geodesic equations numerically. In addition, we had trouble getting Mathematica to efficiently obtain the function \( f(s) \) for the purposes of plotting the surface.

In MATLAB, we first wrote code which allows a user to prescribe the warping function \( r(s) \) as well as some initial data for a geodesic. The user runs the code and it plots images like the following in a MATLAB figure. The user is also able to draw additional geodesics into the figure.

![Figure 1: The figure shows the surface generated by \( r(s) = \log(s + 2) \) with one timelike, one lightlike, and one spacelike geodesic, each emanating from the same initial point.](image)

The code can be summarized as follows.
• Define initial data for a geodesic \( \gamma: r(s), \theta_0, \dot{s}_0, \dot{\theta}_0, \) the time interval \([t_b, t_f]\) on which to solve the geodesic equations with the initial time \(t_0 \in [t_b, t_f]\).

• Solve the geodesic equations using the ode45 numerical differential equation solver in MATLAB to obtain \(s(t)\) and \(\theta(t)\). In addition to solving for \(s(t)\), we had the ODE solver obtain the \(z\)-coordinate \(z(t) = f(s(t))\) along \(\gamma\) using the equation

\[
z'(t) = \sqrt{1 + r'(s)^2} \cdot \frac{ds}{dt}
\]

as part of the system ODEs. This allowed us to bypass the speed problems we were having in Mathematica.

• Find the minimum and maximum \(s\) values \(s_{\text{min}}\) and \(s_{\text{max}}\) along \(\gamma\) to figure out how much of the surface is needed for plotting purposes. Using this data, use the geodesic solver again to find the profile curve \((r(t), 0, f(t))\) (which is a geodesic) over the interval \([s_{\text{min}}, s_{\text{max}}]\).

• MATLAB had a useful function called cylinder that will take a curve and produce numerical coordinates for a surface obtained by rotating that curve around the \(z\)-axis in 3-space.

• Obtain a proper coordinate grid on the surface to illustrate equally spaced \(s\) values. Because this surface is embedded in Minkowski space, it’s useful to see coordinate lines which represent equally spaced coordinate values.

With this, we were able to investigate many interesting surfaces and the behavior of geodesics on them. One interesting example we often looked at was when \(r(s) = \sqrt{1 - s^2}\). The interesting thing about this surface is that near \(s = \pm 1\), it has singularities from/to which the surface is expanding/collapsing at the speed of light. See below.

![Figure 2: Surface defined by \(r(s) = \sqrt{1 - s^2}\) with geodesics of each causal character plotted.](image-url)
After we had a useful MATLAB code written, we eventually discovered that MATLAB has a very useful GUI building tool called GUIDE. With relative ease, GUIDE allows a user to create an interactive GUI with buttons, sliders, user inputs, etc. With this tool, we created a widget that allows a user to do these things without having to dig into our MATLAB code. The following is an image of the widget.

![Figure 3: A draft of the widget created for our project.](image)

**Future IGL Work**

We have three general goals going forward on this IGL project:

- Add more features to the widget and make it easier to use for users who might not be familiar with the mathematics.
- Create animations of the geodesics using actual time as a coordinate, rather than using static images.
- Create visualization tools for geodesics on more general warped product spacetimes.

We would like to include more features into our widget. First, the initial data in our widget are not the most intuitive pieces of data to use. Instead of entering $\dot{s}_0, \dot{\theta}_0$, it would be more intuitive for
the user to enter the speed of the geodesic and some other constant like an initial angle or the initial tangent of the angle from the initial horizontal circle. However, angles are strange in Minkowski space because it becomes necessary to introduce complex angles.

In addition, we would like the widget to include animation capabilities. Since we are studying objects in spacetime, it would be useful to actually use real time as our time coordinate, rather than the $z$-axis of a figure. Animations of the behavior in the tangent space would also be useful because it can illustrate a Minkowski version of Clairaut’s relation from differential geometry.

Finally, one can think of these surfaces of revolution as examples of warped product spaces of the form $(-\mathbb{R}) \times_r S^1$, i.e. universe consisting of a circle which expands and contracts in circumference over time according to the warping function $r$. We are interested in seeing animations of geodesics in warped product spaces of the form $(-\mathbb{R}) \times_r M$ where $M$ is some more complicated Riemannian manifold, for example, the 2-sphere or hyperbolic space.