Existence of Non-Smooth Extremals

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Historical Context

Traditionally solutions to functions which summarize the behavior of dynamical systems are thought to be smooth. Dynamical Systems are defined as systems in which the future state is described by the current state. From past conjectures in SubRiemannian geometry, it is supposed that we do not always have smooth solutions however. Our project focuses on exploring and discovering such counter examples through the study of a dynamical system of linear differential equations.

Introduction

The set of equations being used mimic a charged particle traveling through a non-constant magnetic field. The resulting trajectories which solve this system of equations produce unique curvatures which is vital to the exploration of non-smooth optimal paths. The equations are as follows:

\[
\begin{align*}
x'(t) &= v(t) \\
y'(t) &= u(t) \\
v'(t) &= F(x,y)u(t) \\
u'(t) &= -F(x,y)v(t) \\
F(x,y) &= y(y-x\epsilon) \\
\epsilon &\geq 0
\end{align*}
\]

In order to construct the baseline trajectory the given boundary conditions are necessary. By taking the unit vector from the origin to the starting point and then taking another unit vector from the origin to the ending point, the baseline trajectory by definitions is of length two and non-smooth. If a trajectory which solves the given boundary value problem can be found with length greater than that of the baseline trajectory, then the resulting trajectory is by definition not optimal. As a result the optimal path is the baseline trajectory itself which is by construction non-smooth.

BVP Solutions

In order for the length of the resulting trajectories to be greater than the baseline trajectory, it is necessary to have some closed loops within the solutions of the boundary value problems. Within these loops is enclosed negative area due to the counterclockwise orientation of the path at this specific time. If enough negative area can be produced by a small enough value of epsilon, then the length of the trajectory which solves the BVP will be greater than that of the baseline trajectory. Thus far there have been no loops observed within the resulting trajectories, which can be attributed to a number of reasons.
Figure 1: The trajectory produced by the solutions to the given boundary value problem exhibiting no loops together with the baseline trajectory for comparison.

It is possible that the constrained time interval under which the BVP’s were being solved prevented the exhibition of loops within the resulting trajectories. The interval was therefore later expanded by one second. It is further possible that the initial guess required by MatLab has also prevented the exhibition of loops. The initial guess was therefore changed from a linear set of equations to a more complex set which more closely fit the behavior of the expected solutions. MatLab was unsuccessful in solving the BVP’s with the new set of equations however, and the linear set of equations are therefore being used for the rest of the project. Under the new time interval, there was still a lack of closed loops observed with the resulting solutions. It therefore became necessary to determine if any value of epsilon would produce the negative area associated with the presence of loops.

Integration

In order to determine if any value of epsilon would produce the desired negative area, the function for force which is dependent on epsilon needed to be integrated. Since epsilon is the only variant within the set of differential equations, it is only necessary to integrate over force itself. The integration with given boundary conditions is as follows:

\[
\int_{-1}^{1} \int_{0}^{\sqrt{1+\epsilon^2}} y(y-\epsilon x) \, dy \, dx.
\]

The resulting function however is positive for all values of epsilon between zero and one. From these results, there does not exist any value of epsilon which would provide the negative area needed to have the presence of closed loops within the resulting trajectories.

Figure 2: The integration of the equation for force under the given boundary conditions.

Since it is not possible to obtain any loops with the given function for force, it is necessary to change the current equation. By exploring similar equations such as \(F(x,y) = y(y-x\epsilon)\), it may be possible to produce the needed loops within the resulting trajectories.