The Sphere in Minkowski Space

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Abstract

In the Alexander group, we worked on visualizations of the "unit sphere" in flat space-time. In space-time, or Minkowski space, the "unit sphere" consists of vectors of square norm 1 and -1. This gives the two parts of the sphere in space-time, the deSitter space, and two copies of hyperbolic space. This unit sphere has interesting properties including arc length with imaginary terms which "do the bookkeeping" for how many 'infinities' away one point is from another.

To begin to find a good intuition of what is happening on this "unit sphere" (we will elaborate on the quotations later), we must first look at the "norm" in this space.

\[
< (x, y, t) > = x^2 + y^2 - t^2 \tag{1}
\]

If you have taken any linear algebra, you will probably see immediately why calling this a norm is incorrect. One requirement of a true norm is that it must be a function from the space to \( \mathbb{R}^+ \), the positive reals. This pseudonorm allows for vectors with negative length which is part of what gives this space its intrigue. The motivation for creating such a space is physical; Minkowski space is the space that Albert Einstein's theory of special relativity takes place in. People often call Minkowski space "flat space-time" because it is the solution to the Einstein field equations for zero curvature. We can see that this norm not only fails to produce strictly non-negative values, but also gives:

\[
< (cx, cy, cz) >= c^2 < (x, y, z) > \tag{2}
\]

This also fails as the corresponding inner product, which follows from its failure as a norm, in this case.

So with this in mind, we can begin to go about constructing our unit sphere. First, we have to consider our intuition of the sphere in a set of vectors. When thinking of the elements in any such set, the sphere is defined not just as vectors with norm 1, but in a more fundamental sense, vectors with norm \(|1|\). This provides an informal reason why we should include both the hyperbolic part of our sphere and the deSitter sheet. It is also nice to have the sphere be closed
as a surface, which is why deSitter space alone is not a particularly satisfying sphere (deSitter space is the set of all vectors of "length" 1). To remedy this, we simply add both hyperbolic "bowls". This surface is closed if you think about its behavior towards infinity, so you can hopefully convince yourself that this is a good sphere for all practical purposes. In the spirit of clarity, and thoroughness, we should mention that if we did decide that only deSitter space was our unit sphere, our arc lengths would behave much like euclidean arc lengths in $S^2$, except imaginary.

Figure 1: The Hyperbolic Spaces

Figure 2: The deSitter Space
In figure 1, you can see the hyperbolic bowls, and in figure 2 you can see the deSitter sheet wrapped around them. To illustrate that they converge at infinity, you can note the change in the range of the plots between figure 3 and figure 4.