Creative Blocking

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Abstract

We define creative blocking, which creates one polygon from another by building a block (typically, a square) on the sides and connecting the nearby vertices of these squares. This process is examined in the complex plane and is generalized in three ways. Finally, we list future directions to take the project.

1 Introduction to Creative Blocking

Let $P_0$ be a finite ordered list of points in the plane. The edges of $P_0$ are the directed line segments between consecutive points, including the edge from the last point back to the first point. $P_0$ is not necessarily a polygon since it could represent a shape that is possibly non-convex or even self-intersecting. $P_0$ could also have repeated vertices so edges with length zero might occur. For convenience sake, we will assume that the initial list of points, $P_0$ represents a convex polygon.

Creative blocking is the iterative process that constructs a new list of points, which represents a new shape, $P_{i+1}$, from $P_i$ as follows. A square is erected on the right side of each edge of $P_i$, where “right” is defined as if one were to be walking on the edges of $P_i$ in the given orientation. With respect to the square erected on an edge $e$, the side opposite of $e$ is taken to be an edge of $P_{i+1}$ with the same orientation as $e$. The endpoints of these edges can then be connected to form the closed polygon $P_{i+1}$.

![Figure 1: $P_i$ and $P_{i+1}$, where $P_i$ is an isosceles right triangle oriented counter-clockwise.](image1.png)
Figure 2: $P_0$, where $P_0$ is as in Figure 1. The shaded region represents where the winding number is odd.

2 Creative Blocking in the plane

Given a polygon $P_i$, each edge of $P_i$ can be identified with a vector in the plane. Half of the edges of $P_{i+1}$ are the edges of $P_i$, just translated by some amount. To see how the other half are created, let $u$ and $v$ be two consecutive edges of $P_i$. Recalling that multiplication by $i$ results in a counter-clockwise rotation by $\pi/2$, we see in Figure 3 that the edge of $P_{i+1}$ between $u$ and $v$ is $x = i(u - v)$.

Figure 3: The edge $x = i(u - v)$ of $P_{i+1}$ is created from the edges $u, v$ of $P_i$.

Therefore, Creative Blocking simply specifies how a new edge $x$ will be created, given two edges, $u$ and $v$, of the previous iteration. To this extent, we can generalize the notion of Creative Blocking by somewhat ignoring the geometric interpretation of Creative Blocking, and just specifying a formula for $x = x(u, v)$. Namely, in the original Creative Blocking, $x(u, v) = i(u - v)$. 
3 Scaled rectangle

Fix $r \in \mathbb{R}$. Instead of erecting a square on each edge of $P_t$, we can erect a rectangle such that the ratio of two adjacent sides is $r$. We will refer to this as $r$-scaled Creative Blocking. Note that the original Creative Blocking is 1-scaled Creative Blocking. If $u$ and $v$ are as before, then we see in Figure 4 that $x = ir(u - v)$.

Figure 4: The edge $x = ir(u - v)$ of $P_{t+1}$ is created from the edges $u$, $v$ of $P_t$ under $r$-scaled Creative Blocking.

Figure 5: $P_6$, where $P_0$ is as in Figure 1 using $r$-scaled Creative Blocking with $r = 1/2$. Compare this to Figure 2, where $r = 1$. 
4 Fixed rectangle

Again fix $r \in \mathbb{R}$. We can also erect a rectangle such that the length of two of its sides is $r$. We will refer to this as $r$-fixed creative blocking. Again letting $u$ and $v$ be as before, we see in Figure 6 that

$$x = ir \left[ \frac{u}{|u|} - \frac{v}{|v|} \right].$$

Figure 6: The edge $x$ of $P_{i+1}$ is created from the edges $u, v$ of $P_i$ under $r$-fixed Creative Blocking.

Figure 7: The tenth iteration of $r$-fixed creative blocking on a degenerate triangle with $r = 1$. 
5 Another generalization

As mentioned before, we can specify a formula \( x = x(u,v) \) without a simple geometric interpretation in mind. For example, for a fixed \( r \in \mathbb{R} \), we can let

\[
x = ir \cdot \frac{u - v}{|u - v|}.
\]

We will refer to this as \( r \)-weird creative blocking. Note that the edges \( v \) must be translated to form a closed polygon.

![Figure 8: The tenth iteration of \( r \)-weird creative blocking on an equilateral triangle oriented counterclockwise with \( r = 1/2 \).](image)

6 Future directions

6.1 Building other shapes on the edges

This is perhaps the most obvious way to go. Suppose that a regular \( k \)-gon is erected on each edge in each iteration. Then how will the new edges be connected exactly?

6.2 Polyhedra and higher dimensions

A cube could be erected on each face of a polyhedron and the resulting faces connected. This can likely be generalized to higher dimensions, but perhaps without a good visualization beyond the third dimension.

6.3 Winding numbers

Each point inside a polygon has a winding number associated to it. Let \( P_0 \) be given, and let \( W_i \) be the set of winding numbers that show up for \( P_i \). What does the sequence \( (W_0, W_1, \ldots) \) look like? What sequences \( (W_0, W_1, \ldots) \) are possible?

6.4 Slopes of edges

Let \( u \) and \( v \) be two consecutive edges of \( P_0 \). Then under usual creative blocking, we can see the edges of \( P_1 \) from the multiplication of \( \binom{u}{v} \) and the matrices

\[
\begin{bmatrix}
1 & 0 \\
i & -1
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
i & -i \\
0 & 1
\end{bmatrix}.
\]
These two matrices generate a finite group of order 96. It follows that only a finite set of different slopes of edges can appear in the sequence \((P_0, P_1, \ldots)\). Let \(S_i\) be the set of the slopes of the edges of \(P_i\). What does the sequence \((S_0, S_1, \ldots)\) look like? What sequences \((S_0, S_1, \ldots)\) are possible?

6.5 Multiplicity and order of points

Let the \(k\)th multiplicity of a point \(p\) in a polygon \(P\) be the number of consecutive times that it occurs while appearing for the \(k\)th non-consecutive time. For example, in \(P = (a, b, a, a, a, a)\), the first multiplicity of \(a\) is 1, the second multiplicity of \(a\) is 4, the first multiplicity of \(b\) is 1, and all other multiplicities are 0.

Let the reduced form of a polygon \(P\) be the same list of points as in \(P\), but with every multiplicity reduced to at most 1. We will say that two polygons are visually equivalent if their reduced forms are equal. Is “visually equivalent” an equivalence relation?

Let the semi-reduced form of a polygon \(P\) be the the same list of points as in \(P\), but with every multiplicity reduced to at most 2. With this definition, we state the following conjecture: Let \(P\) be a generalized polygon, with \(P'\) and \(P''\) the reduced and semi-reduced forms respectively, and let \(n\) be a nonnegative integer. Then,

(i) if \(n < 2\), \(P_n\) is visually equivalent to \(P'_n\), and

(ii) if \(n \geq 2\), \(P_n\) is visually equivalent to \(P''_n\) but not to \(P'_n\).