Applications of n-dimensional Integrals:
Random Points, Broken Sticks
and Intersecting Cylinders

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Abstract

This project is part of a broader program to seek out and explore interesting new or little
known applications of n-dimensional integrals. In Fall 2012 we focused on two such applications,
each motivated by a well-known classical problem. The first of these problems is the volume of
the “Steinmetz solid”, the region of intersection of three pairwise perpendicular cylinders of unit
radius depicted in Figure 1 below. We introduced an appropriate notion of an n-dimensional
Steinmetz solid, and we obtained exact formulas for its volume for dimensions up to 5, and
numerical values for dimensions up to 9.

The second problem is the “Broken Stick Problem”, which first appeared in a 19th century
examination at Cambridge University. The problem asks for the probability that the three
pieces obtained by breaking up a stick at two randomly chosen points can form a triangle. In
our project we considered analogous questions for broken sticks with n pieces, and we obtained
both theoretical and numerical results on these questions.

Our team also created animations and interactive modules to help visualize the geometric
aspects of these problems. On November 30, 2012, three members of the team visited Urbana
High School to give a presentation to the Math Club on this research.
1 Introduction

There exist many interesting problems in areas ranging from geometry to probability and statistics, to mathematical biology that lead naturally to \( n \)-dimensional integrals. Such integrals can, in principle, be handled using the same techniques as double and triple integrals and thus are accessible to any student with a strong calculus background. Yet, such applications are rarely found in calculus texts, where the focus is almost exclusively on integrals in two and three dimensions. The only example of a higher-dimensional integration problem that does receive some coverage in traditional calculus texts—typically as a special project/honors level problem—is the computation of the volume of a hypersphere in \( n \) dimension; see, for example, the Discovery Project “Volumes of Hyperspheres” in Stewart’s Calculus text [12].

The project described here is part of a broader program to seek out and explore interesting new or little known applications of \( n \)-dimensional integrals. For this project we focused on two such applications, each motivated by a well-known problem in the familiar two- or three-dimensional setting: The “Intersecting Cylinders Problem”, which asks for the volume of the region of intersection of three pairwise perpendicular cylinders, and the “Broken Stick Problem”, which asks for the probability that the three pieces obtained by breaking up a stick at two randomly chosen points can form a triangle.

For each of these problems, we formulated appropriate \( n \)-dimensional versions of the problem, we investigated these generalizations theoretically and experimentally, and we created animations and interactive modules to help visualize the geometric aspects of the problem.

2 Intersecting Cylinders

2.1 The classical Intersecting Cylinders Problem

Our first project is motivated by the following problem, which can be found in many calculus texts; see, for example, the Discovery Project “Intersections of Three Cylinders” in Stewart [12].

Problem 1. What is the volume of the region of intersection of three pairwise perpendicular cylinders of unit radius?

The region of intersection of the three cylinders is depicted in Figure 1, along with the corresponding region of intersection of two perpendicular cylinders. Both regions are referred to as “Steinmetz solid”, after Charles Proteus Steinmetz, a mathematician and engineer from the early 20th century who is said to have solved the two-cylinder version of this problem in two minutes while at a dinner party [13].

The two-cylinder version of the problem was first studied more than two thousand years by Archimedes [5], who proved that the volume of a two-cylinder Steinmetz solid is \( 16/3 \). The three-cylinder version appears to have been first considered by Moore [9], a physicist who was motivated by applications in crystallography. Moore showed that the volume of a three-cylinder Steinmetz solid is \( 16 - 8\sqrt{2} \), and he also gave formulas for versions of the Steinmetz solid involving more than three cylinders;

The problem has been popularized by Martin Gardner in his Scientific American column [1], and the two-dimensional Steinmetz solid is featured on the cover of one of Gardner’s books [2]. It received further notoriety when Steven Strogatz mentioned it in his New York Times column [13] as an illustration of Archimedes’ slicing method to compute volumes. The problem has its own MathWorld entry [16] and Wikipedia page [17], where more information, and many references, can be found.
Figure 1: The two- and three-cylinder versions of the Steinmetz solid. The figures on the left show the cylinders; the figures on the right show their regions of intersection.

A variety of generalizations and variations of this problem have been studied, including intersections of six and eight cylinders, intersections of cylinders of different radii, and intersections of elliptical cylinders; see the MathWorld article cited above for details and references. However, generalizations to higher dimensions do not seem to have been considered in the literature. Our goal was to formulate and solve—to the extent possible—such generalizations.

2.2 Cylinders in n dimensions

In order to come up with an appropriate n-dimensional analog of the Steinmetz solid, we first need to introduce a notion of a cylinder in n-dimensional space. In the usual three-dimensional space \(\mathbb{R}^3\), a cylinder of radius \(r\) about an axis \(L\) can be described as the set of all points in \(\mathbb{R}^3\) whose distance to \(L\) is at most \(r\).

We now generalize this definition to the \(n\)-dimensional space \(\mathbb{R}^n\). The notion of a line carries
over immediately to $\mathbb{R}^n$: a line in $\mathbb{R}^n$ is the set of points of the form $a + tb$, where $t \in \mathbb{R}$ and $a$ and $b$ are fixed vectors in $\mathbb{R}^n$, with $b$ nonzero. Similarly, the notion of distance between two points extends in a natural way to the $n$-dimensional setting: Given two points $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, the (Euclidean) distance between these points is defined as

$$d(x, y) = |x - y| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$ 

Finally, we define the distance between a point $x$ and a line $L$ in $\mathbb{R}^n$ as the minimal distance between $x$ and a point $y$ on this line:

$$d(x, L) = \min_{y \in L} d(x, y).$$

With these notations, we are ready to define cylinders in $\mathbb{R}^n$:

**Definition 1.** Given a line $L$ in the $n$-dimensional space $\mathbb{R}^n$ and a positive number $r$, we define $C(L, r)$, the 
 **cylinder of radius** $r$ **about the** axis $L$, as the set of all points in $\mathbb{R}^n$ whose distance to $L$ is at most $r$:

$$C(L, r) = \{x \in \mathbb{R}^n : d(x, L) \leq r\}.$$ 

**2.3 The Steinmetz solid in $n$ dimensions**

Having defined $n$-dimensional cylinders, we now generalize the definition of a Steinmetz solid to $n$ dimensions as follows:

**Definition 2.** The $n$-dimensional Steinmetz solid $S_n$ is the intersection of the cylinders of radius 1 about the $n$ coordinate axes in $\mathbb{R}^n$:

$$S_n = C(L_1, 1) \cap \cdots \cap C(L_n, 1),$$

where $L_i$ denotes the $i$-th coordinate axis, i.e., $L_i = \{te_i : t \in \mathbb{R}\}$, with $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, the $i$-th unit basis vector in $\mathbb{R}^n$.

To make this definition more concrete, observe that the distance of a point $(x_1, \ldots, x_n)$ from the $x_1$-axis is $\sqrt{x_2^2 + \cdots + x_n^2}$. Thus, the cylinder $C(L_1, 1)$ is the set of points $(x_1, \ldots, x_n)$ satisfying

$$x_2^2 + x_3^2 + \cdots + x_n^2 \leq 1,$$

and the cylinders $C(L_2, 1), \ldots, C(L_n, 1)$ are given by analogous inequalities, each with one “missing” variable. Hence the Steinmetz solid $S_n$ is the region in $\mathbb{R}^n$ defined by the $n$ constraints

$$\begin{align*}
    x_2^2 + x_3^2 + \cdots + x_n^2 & \leq 1, \\
    x_1^2 + x_3^2 + \cdots + x_n^2 & \leq 1, \\
    \vdots & \\
    x_1^2 + x_2^2 + \cdots + x_{n-1}^2 & \leq 1.
\end{align*}$$

(1)

In the 3-dimensional case, with the variables denoted by $x, y, z$, the conditions (1) reduce to the conditions $x^2 + y^2 \leq 1$, $x^2 + z^2 \leq 1$, and $y^2 + z^2 \leq 1$, which represent ordinary cylinders of unit radius about the $z$-, $y$-, and $x$-axes, respectively. These are precisely the cylinders depicted in Figure 1, whose intersection is the ordinary three-dimensional Steinmetz solid.

In the next case, $n = 4$, the conditions (1) represent four cylinders in 4-dimensional space about the coordinate axes. While we cannot visualize these cylinders and their region of intersection directly, we can do so indirectly via 3-dimensional “slices”. Specifically, if we let the coordinate variables denote by $x, y, z, w$, then fixing a value of $w$ in (1) gives a set of constraints on $x, y, z$ that represents a 3-dimensional “slice” of the 4-dimensional Steinmetz solid. Figure 2 depicts some of these slices in the $xyz$-space as $w$ ranges from $w = 1$ to $w = 0$. 

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2.4 The volume of the \( n \)-dimensional Steinmetz solid

Our main goal is to compute the volume, \( V(S_n) \), of the region \( S_n \) defined by (1). This is the \( n \)-dimensional analog of the problem stated at the beginning of this section. We were able to carry out this computation completely, and obtain an exact formula for the volume, for the first two cases beyond the three-dimensional case, namely the 4- and 5-dimensional cases:

**Theorem 1** (IGL Team). We have

\[
V(S_4) = 48 \left( \frac{\pi}{4} - \frac{1}{\sqrt{2}} \tan^{-1}\sqrt{2} \right), \quad V(S_5) = 256 \left( \frac{\pi}{12} - \frac{1}{\sqrt{2}} \cot^{-1}(2\sqrt{2}) \right).
\]

For higher dimensions we have not (yet) obtained closed formulas for the volumes \( V(S_n) \), but we have used Monte Carlo simulations to evaluate the corresponding integrals numerically. The results are reliable up to dimension \( n = 9 \). Table 1 summarizes our results.
### Table 1: Summary of numerical data and known theoretical formulas for the volumes of Steinmetz solids in various dimensions. Here $S_{3,2}$ denotes the 2-cylinder version of the Steinmetz solid in $\mathbb{R}^3$, and $S_n$ is the $n$-dimensional Steinmetz solid defined above. Observe that, within the range computed, the volume is maximal in dimension $n = 5$.

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Solid</th>
<th>Numerical volume</th>
<th>Exact volume</th>
<th>Credit</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$S_{3,2}$</td>
<td>5.3333</td>
<td>$\frac{16}{3}$</td>
<td>Archimedes (ca. 250 BC)</td>
</tr>
<tr>
<td>3</td>
<td>$S_3$</td>
<td>4.68625</td>
<td>$16 - 8\sqrt{2}$</td>
<td>Moore (1974)</td>
</tr>
<tr>
<td>4</td>
<td>$S_4$</td>
<td>5.27511</td>
<td>$48 \left( \frac{\pi}{4} - \frac{1}{\sqrt{2}} \tan^{-1} \sqrt{2} \right)$</td>
<td>IGL Team (2012)</td>
</tr>
<tr>
<td>5</td>
<td>$S_5$</td>
<td>5.50445</td>
<td>$256 \left( \frac{\pi}{12} - \frac{1}{\sqrt{2}} \cot^{-1}(2\sqrt{2}) \right)$</td>
<td>IGL Team (2012)</td>
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<tr>
<td>6</td>
<td>$S_6$</td>
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<td>9</td>
<td>$S_9$</td>
<td>3.34627</td>
<td></td>
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</table>

3 Broken Sticks

3.1 The classical Broken Stick Problem

The original version of this problem appeared in the mid 19th century as an exam problem at Cambridge University:

**Problem 2.** A rod is marked at random at two points and then divided into three parts at these points; shew that the probability of it being possible to form a triangle with the pieces is $\frac{1}{4}$.

The problem attracted the interest of the 19th century French probabilists Lemoine [6] and Poincaré [11], was further studied in the mid 20th century by Lévy [7], and was popularized by Martin Gardner [3]. It gives rise to the “broken stick model”, an important probabilistic model that arises in areas ranging from biology (MacArthur [8] and Pielou [10]) to finance (Tashman and Frey [15]). The model has been shown to be a good match for a variety of real-world data sets, including intervals between twin births reported in the Champaign-Urbana News-Gazette, intervals between rainy days reported at the airport in North Bay, Ontario, Canada, and intervals between aircraft crashes of U.S. Carriers (Ghent and Hanna [4]).

3.2 The Broken Stick Problem with $n$ pieces: Setup

To obtain an $n$-piece version of this problem, we consider a stick that is broken up at $n - 1$ points chosen at random along its length. We are interested in forming triangles with the resulting $n$
pieces. There are three natural questions one can ask.

**Question 1.** What is the probability that there exist three of the \( n \) pieces that can form a triangle?

**Question 2.** What is the probability that any triple of pieces chosen from the \( n \) pieces can form a triangle?

**Question 3.** What is the probability that a triple of pieces chosen at random from the \( \binom{n}{3} \) triples can form a triangle?

In the case \( n = 3 \) there is one triple of pieces, and all three questions reduce to the original Broken Stick Problem. Thus, the questions represent generalizations of this problem.

For a concrete illustration of these question consider the case \( n = 5 \). In this case, there are \( \binom{5}{3} = 10 \) triples of pieces, each of which may or may not form a triangle. Question 1 is concerned with the probability that at least one of these 10 triples forms a triangle. Question 2 asks for the probability that all 10 triples form a triangle. For Question 3 a triple is chosen “at random” (i.e., each of the 10 triples is equally likely to be drawn), and we are interested in the probability that such a randomly chosen triple forms a triangle.

### 3.3 The Broken Stick Problem with \( n \) pieces: Results

We were able to answer Questions 1 and 2 exactly by proving the following theorem.

**Theorem 2** (IGL Team). Consider a broken stick with \( n \) pieces, obtained by selecting \( n - 1 \) breaking points along the stick at random (i.e., independently and uniformly).

- The probability that there exist three of the \( n \) pieces that can form a triangle is

\[
1 - \prod_{k=2}^{n+1} \frac{k}{F_{k+2} - 1},
\]

where \( F_n \) is the \( n \)-th Fibonacci number.

- The probability that any triple of pieces chosen from the \( n \) pieces can form a triangle is

\[
\frac{n!(n-2)!}{(2n-2)!}
\]

For \( n = 3 \) both formulas reduce to \( \frac{1}{4} \), which is consistent with the solution to the original Broken Stick Problem. For \( n = 4 \) the first formula gives \( \frac{4}{7} \) as the probability that, if a stick is broken into 4 random pieces, at least one triangle can be formed from these pieces.

For Question 3 we have not (yet) obtained an exact formula. However, we have performed numerical simulations that indicate that the probability sought in Question 3 is close to \( \frac{1}{4} \), and approaches \( \frac{1}{4} \) as \( n \to \infty \).

### References


