Illinois Geometry Lab

Apollonian Circle Packing Density

Author:
Joseph Vandehey
Danni Sun
Jason Hempstead
Kaiyue Hou

Faculty Mentor:
Jayadev Athreya

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The very first thing we did was to simplify general Apollonian Circle Packings to the Farey-Ford circle packing. The Farey-Ford packing on the interval [0, 1] is created by circles with center at \((\frac{p}{q}, \frac{1}{q^2})\) and radius \(\frac{1}{q^2}\), \(p, q \in \mathbb{N}, p < q\). We then drew a horizontal line \(y = h\) and calculated what proportion of the line is inside the circles. That is, to find the ratio of the length of the line segments that “cut through” the Farey-Ford circles to the length of the whole line.

Let \(f(h)\) be the measure of the length of these line segments at height \(h\), with \(f_r(h)\) the measure of the length through a circle of radius \(r\). Then we have

\[
f(h) = \sum_{0 \leq \frac{p}{q} < 1, \frac{1}{q^2} \geq h} f_{\frac{1}{q^2}}(h) = \sum_{h \frac{1}{q^2} \geq q \geq 1} (2 \sqrt{h(\frac{1}{q^2} - h)} \cdot \phi(q))
\]

Figure 1 shows that \(\lim_{h \to 0} f(h) \approx 0.955\). In fact, the limit goes to \(\frac{3}{\pi}\).

II. Error Term

We measured how quickly the function \(f(h)\) approaches \(\frac{3}{\pi}\). The error term is

\[
R(h) = |f(h^{-2}) - \frac{3}{\pi}|
\]

Since \(R(h)\) appears to be bounded by \(\frac{1}{h^{3/2}}\), we define \(A(h) = h^{3/2} \cdot R(h)\). The
plots for $R(h)$ and $A(h)$ can be seen in Figure 2 and 3.

III. Möbius Transformation

A Möbius Transformation is a rational function of the form

$$f(z) = \frac{az + b}{cz + d}$$

with $ad - bc \neq 0$. We often think of this transformation as being a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $M_2(\mathbb{C})$ with non-zero determinant. The three transforms

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ -1 & i \end{pmatrix}$$
Figure 4: Mobius transformations applied to a part of the Farey-Ford packing

preserve the original Farey-Ford packing. These three transformation matrices are such that
the first translates the packing one unit to the right (its inverse translates one unit to the
left), the second reflects through a circle, and the third maps the 3 major points (0, i, and
∞) cyclically: (i ⇒ ∞, ∞ ⇒ 0, 0 ⇒ i). Using these transformations, we try to create the
'full' packing between y = 0 and y = 1 to arbitrary precision (see Figure 5). Using other
matrices, we generated even more Apollonian Circle Packings (see Figure 4).

IV. More Transformations and Generalizations to 3 Dimensions

We also worked on generating a sphere packing in three dimensions using inversion. Given
four mutually tangent spheres, there is a dual sphere that passes through all of their points
of tangency. Moreover, a sphere packing is preserved by inversion through one of its dual
spheres. Initially there are 3 identical spheres on xy-plane with centers at (0, 0, 1/2), (1, 0, 1/2)
and (1/2, \sqrt{3}/2, 1/2), all with radius 1/2. By continually inverting through different dual spheres,
we managed to create a sphere packing on the xy-plane.

We have also investigated the analog of $f(h)$ in the three-dimensional case. We use curvatures (one over the radius) instead of radius for convenience. According to Soddy’s Theorem, if five spheres are tangent to each other at distinct points, and the spheres have curvatures $k_i$ (for $i = 1, \ldots, 5$), then $3(k_1^2 + k_2^2 + k_3^2 + k_4^2 + k_5^2) = (k_1 + k_2 + k_3 + k_4 + k_5)^2$. Given four initial spheres, there are two possible mutually tangent “fifth spheres.” Using these curvatures we can calculate $f(h)$ without the need to determine where all the spheres are located.