k-DEPENDENCE, DISJOINT MATCHINGS, AND AN EXTENSION OF A THEOREM OF FAVARON

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Abstract. A vertex set \( D \) in a graph \( G \) is \( k \)-dependent if \( G[D] \) has maximum degree at most \( k - 1 \), and \( k \)-dominating if every vertex outside \( D \) has at least \( k \) neighbors in \( D \). Favaron [2] proved that if \( D \) is a \( k \)-dependent set maximizing the quantity \( k|D| - |E(G[D])| \), then \( D \) is \( k \)-dominating. We extend this result, showing that such sets satisfy a stronger property: given any ordering \( < \) of \( V(G) - D \), there is a \( k \)-edge-chromatic subgraph of \( G \) in which every vertex \( v \) outside \( D \) has degree at least \( k - d'(v) \), where \( d'(v) \) is the number of earlier neighbors of \( v \) in \( V(G) - D \). Since any vertex outside \( D \) may be taken as a minimal element of \( < \), this implies that \( D \) is \( k \)-dominating.

1. Introduction

A vertex set \( D \) in a graph \( G \) is independent if the induced subgraph \( G[D] \) has no edges. A vertex set is dominating if every vertex of \( G \) either lies in the set, or has a neighbor in the set. Ore [9] observed that any maximal independent set is also a dominating set: by the maximality of the independent set, every vertex outside the set must have a neighbor in the set. Thus, \( \gamma(G) \leq \alpha(G) \) for any graph \( G \), where \( \gamma(G) \) is the size of a smallest dominating set and \( \alpha(G) \) is the size of a largest independent set.

Fink and Jacobson [3, 4] generalized the notions of independence and domination as follows:

Definition 1. For positive integers \( k \), a vertex set \( D \subset V(G) \) is \( k \)-dependent if the induced subgraph \( G[D] \) has maximum degree at most \( k - 1 \). A vertex set \( D \) is \( k \)-dominating if \( |N(v) \cap D| \geq k \) for all \( v \in V(G) - D \).

Fink and Jacobson posed the following question: letting \( \gamma_k(G) \) denote the size of a smallest \( k \)-dominating set in \( G \) and letting \( \alpha_k(G) \) denote the size of a largest \( k \)-dependent set in \( G \), is it true that \( \gamma_k(G) \leq \alpha_k(G) \) for all \( k \)? Setting \( k = 1 \) yields the original inequality \( \gamma(G) \leq \alpha(G) \). However, for \( k > 1 \) it is no longer true that every maximal \( k \)-dependent set is \( k \)-dominating. Favaron [2] answered the question of Fink and Jacobson, using a different notion of “optimality” for \( k \)-dependent sets:

Theorem 2 (Favaron [2]). If \( D \) is a \( k \)-dependent set maximizing the quantity \( k|D| - |E(G[D])| \) (over all \( k \)-dependent sets), then \( D \) is a \( k \)-dominating set.

Since any set of at most \( k \) vertices is a \( k \)-dependent set, it follows that every graph has a set of vertices which is both \( k \)-dependent and \( k \)-dominating, which yields \( \gamma_k(G) \leq \alpha_k(G) \). Theorem 2 motivates the following definition.

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Definition 3. For any $k$-dependent set $D$, define $\phi_k(D) = k|D| - |E(G[D])|$. A $k$-optimal set is a $k$-dependent set maximizing $\phi_k$.

The notation $\phi_k(D)$ is borrowed from the survey paper [1]. Our goal in this paper is to extend Theorem 2 by proving that $k$-optimal sets satisfy a property stronger than $k$-domination:

Theorem 4. Let $D$ be a $k$-optimal set in a graph $G$, let $X = V(G) - D$, and let $H$ be the maximal bipartite subgraph of $G$ with partite sets $D$ and $X$. If $<$ is an ordering of $X$, then $H$ has a $k$-edge-chromatic subgraph $M$ such that $d_M(v) + d^-(v) \geq k$ for all $v \in X$, where $d^-(v) = |\{w \in N(v) \cap X : w < v\}|$.

In particular, since we can take any vertex $v \in V(G) - D$ to be minimal in $<$, Theorem 4 implies that any $k$-optimal set is $k$-dominating.

2. Extending Lebensold’s Theorem

Lebensold [7] proved the following generalization of Hall’s Theorem [6]. As Brualdi observed in his review of [7], the theorem is equivalent to a theorem of Fulkerson [5] concerning disjoint permutations in 0,1-matrices. An alternative proof of the theorem, using matroid theory, is due to Murty [8].

Theorem 5 (Lebensold [7]). An $X,D$-bigraph has $k$ disjoint matchings from $X$ into $D$, each saturating $X$, if and only if

$$\sum_{v \in D} \min\{k, |N(v) \cap X_0|\} \geq k|X_0|$$

for every subset $X_0 \subset X$.

We extend the theorem to find necessary and sufficient conditions for the existence of a $k$-edge-chromatic subgraph in which the vertices of $X$ are allowed to have different degrees.

Lemma 6. Let $H$ be an $X,D$-bigraph, and write $X = \{v_1, \ldots, v_t\}$. Let $k$ be a positive integer and let $d_1, \ldots, d_t$ be nonnegative integers with all $d_i \leq k$. The following are equivalent:

1) $H$ has a $k$-edge-chromatic subgraph $M$ such that $d_M(v_i) \geq d_i$ for all $i$;
2) For every subset $X_0 \subset X$,

$$\sum_{v \in D} \min\{k, |N(v) \cap X_0|\} \geq \sum_{v_i \in X_0} d_i.$$  

Theorem 5 is the special case of Lemma 6 obtained when all $d_i = k$. We prove Lemma 6 using Theorem 5, so Theorem 5 is self-strengthening in this sense.

Proof. For each $i$, let $D_i$ be a set of size $k - d_i$, with all sets $D_i$ disjoint from each other and disjoint from $V(H)$, and let $D' = D \cup D_1 \cup \cdots \cup D_t$. Let $H'$ be the $X,D'$-bigraph obtained from $H$ by making the vertices in $D_i$ adjacent only to $v_i$. Consider the following two statements:

1) $H'$ has $k$ edge-disjoint matchings, each saturating $X$;
2) For every subset $X_0 \subset X$,

$$\sum_{v \in D'} \min\{k, |N(v) \cap X_0|\} \geq k|X_0|.$$
By Theorem 5, (1′) is equivalent to (2′). We prove that (1) is equivalent to (1′) and (2) is equivalent to (2′).

If $M_1, \ldots, M_k$ are edge-disjoint matchings in $H'$ each saturating $X$, then their restriction to $H$ yields a $k$-edge-chromatic subgraph $M$ of $H$ with each $d_M(v_i) \geq d_i$. Conversely, any such subgraph of $H$ can be extended to $k$ edge-disjoint matchings in $H'$. Thus, (1) is equivalent to (1′).

Elements of $D_t$ each contribute 1 to the sum in (2′) when $v_i \in X_0$, and contribute 0 otherwise. This yields
\[ \sum_{v \in D'} \min\{k, |N(v) \cap X_0|\} = \sum_{v_i \in X_0} (k - d_i) + \sum_{v \in D} \min\{k, |N(v) \cap X_0|\}, \]
so (2) is equivalent to (2′).

3. PROOF OF THEOREM 4

We first define an operation that we will need in order to prove Theorem 4. The definition is based on Favaron’s proof of Theorem 2. In this section, when $T$ is a vertex set and $v$ is a vertex, we often write $N_T(v)$ for $N(v) \cap T$, and likewise for $d_T(v)$ and $d_T(v)$.

**Definition 7.** When $D$ is a $k$-dependent set and $v$ is a vertex of $V(G) - D$ such that $|N_D(v)| < k$, we define the set $D \oplus v$ as follows. Let $A = \{w \in N_D(v): d_X(w) = k - 1\}$, and let $S$ be a maximal independent set in $A$. We define $D \oplus v$ to be the set $(D - S) \cup \{v\}$.

**Definition 8.** Suppose that $D$ is a $k$-dependent set, $Z$ is a set disjoint from $D$, and $<$ is an order on $Z$ such that $d_Z(v) + d_D(v) < k$ for all $v \in Z$. We define the set $D \oplus Z$ as follows: let $z_1, \ldots, z_\ell$ be the vertices of $Z$, written in order according to $<$. Let $D_0 = D$, and for $i \in [\ell]$, let $D_i = D_{i-1} \oplus z_i$. The set $D \oplus Z$ is defined as $D_\ell$.

Strictly speaking, the definition of $\oplus <$ depends on the choice of the independent set $S$ when we apply $\oplus$; however, these choices can be made arbitrarily. The following lemma can be directly extracted from Favaron’s proof of Theorem 2.

**Lemma 9** (Favaron). If $D$ is $k$-dependent and $v$ is a vertex of $V(G) - D$ with $|N_D(v)| < k$, then $D \oplus v$ is a $k$-dependent set with $\phi_k(D \oplus v) = \phi_k(D) + k - |N_D(v)|$.

**Corollary 10.** If $D$ is $k$-dependent, $Z$ is a set disjoint from $D$, and $<$ is an ordering on $Z$ such that $d_Z(v) + |N_D(v)| < k$ for all $v \in Z$, then $D \oplus Z$ is a $k$-dependent set with
\[ \phi_k(D \oplus Z) \geq \phi_k(D) + k |Z| - \sum_{v \in Z} (|N_D(v)| + d_Z(v)). \]

**Proof.** Let $z_1, \ldots, z_\ell$ be the vertices of $Z$ written in order according to $<$, and let $D_0, D_1, \ldots, D_\ell$ be as in Definition 8. For each $i$, we have $D_i \subset D \cup \{v_1, \ldots, v_i\}$, which yields
\[ |N(v_i) \cap D_{i-1}| \leq |N_D(v)| + d_Z(v) < k. \]
Thus, repeatedly applying Lemma 9 yields
\[ \phi_k(D \oplus Z) = \phi_k(D) + k |Z| - \sum_{v \in Z} (|N_D(v)| + d_Z(v)) \geq \phi_k(D) + k |Z| - \sum_{v \in Z} (|N_D(v)| + d_Z(v)). \]
Figure 1. Relationship among the sets $D$, $X$, $X_0$, $B$, $E$, $A$, and $C$.

Proof of Theorem 4. Let $D$ be a $k$-dependent set, let $X = V(G) - D$, and let $<$ be an ordering on $X$. Assuming that there is no $k$-edge-chromatic subgraph with the desired properties, we construct a $k$-dependent set $D'$ with $\phi_k(D') > \phi_k(D)$.

Let $v_1, \ldots, v_t$ be the vertices of $X$, written in order according to $<$. Since there is no $k$-edge-chromatic subgraph with the desired properties, applying Lemma 6 with $d_i = \max\{0, k - d^-(v)\}$ shows that there is a set $X_0 \subset X$ such that

$$\sum_{v \in D} \min\{k, |N(v) \cap X_0|\} < \sum_{v \in X_0} \max\{0, k - d^-(v)\}.$$

We may assume that $d^-(v) \leq k$ for all $v \in X_0$, since vertices with $d^-(v) > k$ may be removed from $X_0$ without causing the above inequality to fail. This gives the simpler inequality

$$\sum_{v \in D} \min\{k, |N(v) \cap X_0|\} < k |X_0| - \sum_{v \in X_0} d^-(v).$$

Define sets $B, E, A, C$ by

$$B = \{v \in D: |N(v) \cap X_0| \leq k - 1\},$$
$$E = \{v \in X_0: |N(v) \cap B| + d^-(v) \leq k - 1\},$$
$$A = D - B,$$
$$C = X_0 - E.$$

The relationship among the various sets is illustrated in Figure 1. Let $D' = B \oplus_c E$. We claim that $\phi_k(D') > \phi_k(D)$. By Corollary 10, $D'$ is $k$-dependent and

$$\phi_k(D') \geq \phi_k(B) + k |E| - \sum_{v \in E} (|N_B(v)| + d^-(v)).$$

Now, observe that

$$\sum_{v \in D} \min\{k, |N_{X_0}(v)|\} = k |A| + \sum_{v \in B} |N_{X_0}(v)|.$$

Thus, from (1),

$$k |A| + \sum_{v \in B} |N_{X_0}(v)| < k |X_0| - \sum_{v \in X_0} d^-(v).$$

Counting the edges incident to $B$ from the endpoints in $X_0$ yields

$$\sum_{v \in B} |N_{X_0}(v)| = \sum_{v \in C} |N_B(v)| + \sum_{v \in E} |N_B(v)| \geq k |C| - \sum_{v \in C} d^-(v) + \sum_{v \in E} |N_B(v)|,$$
where we have used the fact that $|N_B(v)| + d^-(v) \geq k$ for $v \in C$. Therefore, (2) yields
\[ k |A| + k |C| - \sum_{v \in C} d^-(v) + \sum_{v \in E} |N_B(v)| < k |X_0| - \sum_{v \in X_0} d^-(v), \]
which rearranges to
\[ 0 < -k |A| + k |E| - \sum_{v \in E} (|N_B(v)| + d^-(v)), \]
using twice the fact that $X_0 - C = E$. Since $\phi_k(B) \geq \phi_k(D) - k |A|$, applying Corollary 10 yields
\[ \phi_k(D') \geq \phi_k(D) - k |A| + k |E| - \sum_{v \in E} (|N_B(v)| + d^-(v)) > \phi_k(D). \]
Thus, when $D$ is $k$-optimal, a $k$-edge-chromatic subgraph with the desired properties exists.

\[ \square \]

References