This summer I was part of the group REGS\textsuperscript{1} under Professor West. We met for an hour and a half in the morning and three hours in the afternoon for three days a week. For the first half of the morning sessions were devoted to presentations about open problems. As the summer progressed, this changed to presentations about results and time to do research. The afternoon sessions were always devoted to research. I worked primarily on the two problems described below.

**Paintability of Complete Multipartite Graphs with Part Size Three**

This summer the open problems Thomas Mahoney presented related to graph paintability. A proper coloring of a graph $G$ is a function $f : V(G) \to \mathbb{N}$ such that whenever $uv \in E(G)$ we have $f(u) \neq f(v)$. The chromatic number of $G$, denoted $\chi(G)$, is the least $k$ such that there is a proper coloring $f$ of $G$ with $f : V(G) \to \{1, \ldots, k\}$. Choosability is a generalization of this. We start with a list assignment $L : V(G) \to 2^{\mathbb{N}}$. An $L$-coloring of $G$ is a proper coloring $f : V(G) \to \mathbb{N}$ with the additional restriction that $f(v) \in L(v)$ for all $v \in V(G)$. The choice number of $G$, denoted $\chi_\ell(G)$, is the least $k$ such that $G$ has an $L$-coloring for every list assignment $L$ with $\min_{v \in V(G)}|L(v)| \geq k$.

The Marker/Remover game is played on a graph $G$ with a token assignment $f$ giving each $v \in V(G)$ a nonnegative number of tokens. On each round Marker marks a subset $M$ of the remaining vertices, which uses up a token on each vertex in $M$. Remover deletes from the graph an independent subset of vertices in $M$. Marker wins by marking a vertex that has no tokens. Remover wins if the entire graph is removed. The paint number of a graph $G$, denoted $\chi_p(G)$ is the least $k$ such that Remover has a winning strategy when $f(v) = k$ for all $v \in V(G)$. The problem of the Marker/Remover game may be viewed as a generalization of choosability. We may view Marker’s selections as giving an on-line list assignment from which Remover must make colorings. Using this view we see that $\chi_\ell(G) \leq \chi_p(G)$.

Let $K_{3\times k}$ denote the complete $k$-partite graph whose parts all have size three. Kierstead proved that $\chi_p(K_{3\times k}) = \lceil (4k - 1)/3 \rceil$. The problem I worked on was to determine $\chi_p(K_{3\times k})$.

I attempted two methods of determining $\chi_p(K_{3\times k})$.

In the first method, I looked to improve a theorem of Kozik, Micek and Zhu that says that $\chi_p(K_{3\times k}) \leq \frac{3}{2}k$. Their result is based on a lemma that applies to complete multipartite graphs with part size at most three. My attempts to focus on complete multipartite graphs with part size three and thereby improve their result were unsuccessful.

In the second method, I worked with Thomas Mahoney to directly find Marker or Remover strategies on $K_{3\times k}$. I focused on Marker strategies while he focused on Remover strategies to counter. After several iterations we found a Remover strategy that beat every Marker strategy I attempted. In our trials on small graphs ($k \leq 4, 7, 10$ chosen based on properties of the Kierstead bound) this Remover strategy resulted in the Kierstead bound. An attempt to test this strategy against

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all Marker strategies was made using a program written by Gregory Puleo. The computational complexity made this unfeasible with our computing resources. Additionally we were unable to prove that this strategy would always result in a win for Remover.

Improper Choosability of Balanced Bipartite Graphs

The open problem I presented was related to improper choosability of graphs.

We say a graph $G$ if $(k, d)$-colorable if there is a function $f : V(G) \rightarrow \{1, \ldots, k\}$ such that the graph induced on $f^{-1}(i)$ has maximum degree at most $d$ for $1 \leq i \leq k$. A graph $G$ is $(k, d)^*$-choosable if for every list assignment $L$ with $\min v \in V(G)|L(v)| \geq k$ there is a function $f : V(G) \rightarrow \bigcup v \in V(G) L(v)$ with $f(v) \in L(v)$ for all $v \in V(G)$ such that the graph induced on $f^{-1}(i)$ has maximum degree at most $d$ for all $i \in \bigcup v \in V(G) L(v)$. For a graph $G$ and fixed $d$ the minimum $k$ such $G$ is $(k, d)^*$-choosable is denoted $\chi^*(G, d)$.

Erős, Rubin, and Taylor showed that if $m \geq \binom{2k-1}{k}$ then $K_{m,m}$ is not $(k, 0)^*$-choosable. Additionally Alon showed that $k > \log_2(m)$ then $K_{m,m}$ is $(k, 0)^*$-choosable. From this we find that $\chi^*(K_{m,m}, 0)$ is on the order of $\log_d(m)$.

I worked with Jennifer Wise on the project with some additional discussion with visiting professor Kevin Milans.

Using an argument stemming from the one made by Erős, Rubin, and Taylor, we showed that if $m \geq (d + 1) \binom{2k-1}{k}$ then $K_{m,m}$ is not $(k, d)^*$-choosable. This gives that $\chi^*(K_{m,m}, d)$ is at least on the order of $\log_2(m)$. While we were able to show that $\chi^*(K_{m,m}, d) \leq \log_2(m) - \log_2(d)$, we were not able to get bounds that matched in order.

Using a different list assignment and we inductively showed that if $m \geq \binom{2(d+1)}{d+1}$ then $K_{m,m}$ is not $(2, d)^*$-choosable. We believe that our argument would generalize to $k > 3$. However the resulting bound is asymptotically worse than the bound from generalizing the result from Erős, Rubin, and Taylor. For this reason, we did not pursue it.

Finally, in response to a question from Kevin, we showed that if $d = m/t$ and $k \geq 2t - 1$ then $K_{m,m}$ is $(k, d)^*$-choosable. This agrees with intuition that if the allowed degree in each color class is high relative to the number of vertices then lists need not be large to guarantee a desirable coloring.

References


