The focus of this project is on the development of a unifying, categorical framework for complete deduction for coalgebra. From an algebraic perspective, deduction systems have been studied extensively by mathematicians and computer scientists. However, their investigations from a coalgebraic point of view have been more modest. Nonetheless, work by Roșu and others using a category theoretic approach indicates a great potential for dualization, and thus, for applications to coalgebra.

Combining notions and techniques from category theory, state-based computation, and logic \[5\], coalgebra aims to be the mathematics of computational dynamics \[4\]. Coalgebras consist of a state space, say \(X\), and a transition map of the form \(\alpha_X : X \rightarrow F(X)\), where \(F\) is an endofunctor of a category \(\mathcal{C}\) (usually the category of sets). Coalgebras, which are the duals of algebras, turn out to be suitable models of general dynamical systems and of infinite data types, and satisfy the powerful proof principle of coinduction, first introduced by Aczel in his seminal work on non-well-founded sets \[1\]. Even though coalgebra is still in its infancy, it has already led to interesting connections between seemingly unrelated fields such as differential equations and automata theory \[9\], as well as provided a new perspective on the study of classical mathematical structures such as metric spaces and partial orders \[10\]. Our research aims to expand our understanding of coalgebras as they relate to deduction systems.

Mathematicians have been interested in the definitional power of equations for a long time. One of the earliest results characterizing a class of algebras as one containing exactly all algebras which satisfy a given set of equations was obtained by Birkhoff in 1935 \[3\]. Birkhoff’s variety theorem states that given a signature \(\Sigma\), a class \(\mathcal{K}\) of \(\Sigma\)-algebras is closed under homomorphic images, subalgebras, and products—such a class is called a variety—if and only if \(\mathcal{K}\) is the collection of all \(\Sigma\)-algebras satisfying some set of equations. The classical notion of \(\Sigma\)-algebra easily generalizes to the

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category theoretic notion of $F$-algebra, where $F$ is an endofunctor. This abstraction to the categorical framework has yielded many Birkhoff-like axiomatizability results, some of which cover modern approaches to equational logics and first-order logic [2, 6, 7]. Moreover, it has also led to the dual notion of $F$-coalgebras, and, naturally, to the question of whether Birkhoff-like theorems could be dualized to obtain similar results for coalgebra. In [7], Grigore Roșu presented a categorical framework using inclusion systems—an alternative to factorization systems—that generalizes equational logic and proved several axiomatizability results in the style of Birkhoff. In this framework, models are objects, equations are special epimorphisms, and satisfaction is injectivity.

A natural next step is to investigate complete deduction systems for coalgebra within the same general framework. This is the core of our research. Some ideas in this direction were introduced by Roșu in the 1990’s and presented in [8] where a complete categorical four-rule deduction system for equational logic was given. Inspired by that work, we introduce the Projective Deduction System and prove it sound with respect to coequational satisfaction abstracted as projectivity as well as complete, but only for finite coequations.

To emphasize the simplicity and generality of our approach, we point out that everything occurs in a single category $\mathcal{C}$, which has a factorization system $⟨\mathcal{E}, \mathcal{M}⟩$.

**Definition 1.** $⟨\mathcal{E}, \mathcal{M}⟩$ is a factorization system for a category $\mathcal{C}$ provided that

1. $\mathcal{E}$ and $\mathcal{M}$ are subcategories of epimorphisms and monomorphisms, respectively, in $\mathcal{C}$,

2. all isomorphisms in $\mathcal{C}$ are both in $\mathcal{E}$ and $\mathcal{M}$, and

3. every morphism $f$ in $\mathcal{C}$ can be factored as $m \circ e$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, uniquely up to isomorphism, that is, if $f = m' \circ e'$ is another factorization of $f$, then there is a unique isomorphism $\alpha$ such that $\alpha \circ e = e'$ and $m' \circ \alpha = m$:
The following lemma is a direct consequence of the definition of factorization systems; however, it is one of their most important features:

**Lemma 1 (Diagonal-fill).** Let $\langle \mathcal{E}, \mathcal{M} \rangle$ be a factorization system for $\mathcal{C}$. Let $f, g \in \mathcal{C}$, $e \in \mathcal{E}$ and $m \in \mathcal{M}$. If $m \circ f = g \circ e$, then there is a unique $h \in \mathcal{C}$ such that $h \circ e = f$ and $m \circ h = g$:

\[
\begin{array}{ccc}
\bullet & \overset{e}{\rightarrow} & \bullet \\
\downarrow f & & \downarrow g \\
\bullet & \overset{h}{\leftarrow} & \bullet \\
\downarrow m & & \downarrow m
\end{array}
\]

From this lemma, we have the following proposition.

**Proposition 1.** Let $\mathcal{C}$ be a category with a factorization system $\langle \mathcal{E}, \mathcal{M} \rangle$. Let $e \in \mathcal{E}$ and $m \in \mathcal{M}$. If the following diagram

\[
\begin{array}{ccc}
A & \overset{e}{\rightarrow} & B \\
1_A \downarrow & & \downarrow m \\
A & \overset{f}{\rightarrow} & C
\end{array}
\]

commutes, then $e$ is an isomorphism and $f \in \mathcal{M}$.

The previous result can be used to prove:

**Proposition 2.** Let $\langle \mathcal{E}, \mathcal{M} \rangle$ be a factorization system for a category $\mathcal{C}$, then $\mathcal{M}$ is closed under the formation of pullbacks.

The following is crucial for the completeness of our deduction system.

**Proposition 3.** Let $\langle \mathcal{E}, \mathcal{M} \rangle$ be a factorization system for a complete category $\mathcal{C}$. If $X \in \mathcal{C}$, then the slice category $\mathcal{M}/X$ is complete.

When $\mathcal{C}$ is additionally $\mathcal{M}$-well-powered, limits in $\mathcal{M}/X$ exist for large diagrams as well, just take the limit of any representative set $D' \subseteq D$.

**Definition 2.** $(m_D : X_D \rightarrow X, \{\pi_i : m_D \rightarrow m_i\}_{i \in I})$ denotes the limit of $D \subseteq \mathcal{M}/X$.

In the remainder of this report we assume that $\mathcal{C}$ is a complete category that admits a factorization system $\langle \mathcal{E}, \mathcal{M} \rangle$ and is $\mathcal{M}$-well-powered.
Definition 3. Given an object \( A \) in \( C \) and a morphism \( m : \bullet \rightarrow X \) in \( \mathcal{M} \), then \( A \) satisfies \( m \) if \( A \) is \( \{ m \} \)-projective. We denote this by \( A \models m \) and extend it to \( M \models m \) for any class \( M \subseteq \mathcal{M} \).

Now we introduce a deduction system by which we can derive monomorphisms in \( \mathcal{M} \).

Definition 4. Let \( M \subseteq \mathcal{M} \). The Projective Deduction System consists of one axiom, Identity, and three deduction rules: Pullback, Extension, and \( M \)-Pullback.

Identity: \[ X \xrightarrow{1_X} X \] Pullback: \[ \begin{array}{c} m_1 \\ m_2 \end{array} \xleftarrow{m} \] Extension: \[ \begin{array}{c} m_1 \\ m \end{array} \xrightarrow{m_1} \] \( M \)-Pullback: \[ \begin{array}{c} m \\ m_f \end{array} \xrightarrow{m} \]

The previous propositions guarantee the closure of our deduction system.

Proposition 4. Let \( M \subseteq \mathcal{M} \) and \( m \in C \). Then, \( M \vdash m \) implies \( m \in \mathcal{M} \).

Definition 5. If \( M \vdash m \) and \( m \) has codomain \( X \), then \( m \) is called an \( X \)-derivation of \( M \). Let \( \mathcal{D}_X(M) \) denote the full subcategory of \( \mathcal{M}/X \) of \( X \)-derivations of \( M \).

Proposition 5. \( \mathcal{D}_X(M) \) is downward-directed.

Now we can show that the Projective Deduction System is sound.

Theorem 1. Let \( M \subseteq \mathcal{M} \) and \( m \in C \). Then, \( M \vdash m \) implies \( M \models m \).
From now on, coequations refer to morphisms in \( \mathcal{M} \) and \( M \subseteq \mathcal{M} \) is a class of coequations with \( \mathcal{M} \)-injective codomains, denoted by \( I \). (From the context, it should be clear when \( I \) is an \( \mathcal{M} \)-injective codomain or a domain of a diagram.)

**Definition 6.** \( D \subseteq \mathcal{M}/X \) is closed under \( M \)-Pullback if for any morphisms \( m : \bullet \rightarrow I \) in \( M \) and \( f : X \rightarrow I \), \( m_f : \bullet \rightarrow X \) belongs to \( D \).

Using the notation of Definition 2, we have

**Lemma 2.** If \( D \subseteq \mathcal{M}/X \) is closed under \( M \)-Pullback, then \( X_D \models M \).

With the notation of Definition 5,

**Proposition 6.** Let \( m : \bullet \rightarrow X \) be a coequation. Then, \( M \models m \) if and only if \( X_{\mathcal{D}X(M)} \models m \).

One would not expect a deduction system to be complete for satisfaction as projectivity without some kind of finiteness. The following suffices for our purposes:

**Definition 7.** The coequation \( m : \bullet \rightarrow X \) is called finite if for each downward-directed diagram \( \mathcal{D} : (I, \leq) \rightarrow \mathcal{M}/X \), each limit cone \( (\lambda : L \rightarrow X, \{ \pi_i : \lambda \rightarrow m_i \}_{i \in I}) \), and each morphism \( f : \lambda \rightarrow m \) in \( \mathcal{M}/X \), there is an \( i \in I \) such that \( f \) factorizes through \( \pi_i \), that is, \( f = g \circ \pi_i \) for some \( g : m_i \rightarrow m \) in \( \mathcal{M}/X \).

The results above yield the completeness of the Projective Deduction System:

**Theorem 2.** Let \( m : \bullet \rightarrow X \) be a finite coequation. Then, \( M \models m \) implies \( M \vdash m \).

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References


