1. Introduction

During the summer and under the advising of Dr. Reznick I studied canonical forms for polynomials and the method of apolarity. Informally, the method of apolarity provides a way of proving that a certain way of writing a generic homogeneous polynomial is in fact a canonical form. Using the method of apolarity it can be proved, for instance, that a generic quaternary cubic can be written as the sum of 5 cubes of linear quaternary forms. The method can also be used to show that certain decompositions are in general not possible. For example (Clebsch): a ternary quartic cannot in general be written as the sum of 5 fourth powers. The following report is mainly based on the paper *Apolarity and Canonical Forms for Homogeneous Polynomials* by Richard Ehrenborg and Gian-Carlo Rota.

2. Preliminaries

Let $H_d(\mathbb{C}^n)$ denote the vector space of homogeneous polynomials in $n$ variables of degree $d$. An element $p(x_1, \ldots, x_n) \in H_d(\mathbb{C}^n)$ will be called a form of degree $d$. Notice that $\dim(H_d(\mathbb{C}^n)) = \binom{n+d-1}{d}$. An element $p(x_1, \ldots, x_n) \in H_d(\mathbb{C}^n)$ can be written as

$$p(x) = \sum_{|i|=d} \binom{d}{i} a_i x^i$$

where

$$\binom{d}{i} = \frac{d!}{i_1! i_2! \cdots i_n!} \quad a_i = a_{i_1, i_2, \ldots, i_n} \quad x^i = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$$

$x = (x_1, \ldots, x_n)$

$|i| = i_1 + \cdots + i_n$

$i! = i_1! i_2! \cdots i_n!$

3. The Apolar Form

Let $V = \mathbb{C}[x_1, \ldots, x_n]$ so $V = \bigoplus_{d \geq 0} H_d(\mathbb{C}^n)$. To define the apolar form we will use the dual space $H_d(\mathbb{C}^n)^*$ of $H_d(\mathbb{C}^n)$. Let's use the notation $H_d^*(\mathbb{C}^n) := H_d(\mathbb{C}^n)^*$. The space $H_d^*(\mathbb{C}^n)$ is naturally isomorphic to $H_d(\mathbb{C}^n)$ so an element in $H_d^*(\mathbb{C}^n)$ is a homogeneous polynomial of degree $d$ and we will use the $n$ variables $u_1, \ldots, u_n$ whenever the polynomial in matter belongs to $H_d^*(\mathbb{C}^n)$. As before, $V^* = \bigoplus_{d \geq 0} H_d^*(\mathbb{C}^n)$, and by the convention just mentioned, $V^* = \mathbb{C}[u_1, \ldots, u_n]$.

The apolar form $[\cdot | \cdot] : V^* \times V \to \mathbb{C}$ is defined by

$$[u^i|x^j] = i! \delta_{ij}$$
and is extended by linearity.

In particular, if \( p \in H_d(\mathbb{C}^n) \), \( q \in H^*_d(\mathbb{C}^n) \) with \( p(x) = \sum_{|i|=d} \binom{d}{i} a_i x^i \) and \( q(u) = \sum_{|i|=d} \binom{d}{i} b_i u^i \) the apolar form becomes

\[
[p|q] = \sum_{|i|=d} i! a_i b_i
\]

In this case, if we forget about the dual space and think of the bilinear form as being defined just in the space \( H_d(\mathbb{C}^n) \), what we get is real inner product in \( H_d(\mathbb{C}^n) \).

**Definition 1.** Let \( f(x) \) be a form of degree \( r \), and let \( g(u) \) be a dual form of degree \( d \). If \( r \leq d \) then \( f(x) \) is apolar to \( g(u) \) whenever

\[
[g(u)|h(x)f(x)] = 0
\]

for all forms \( h(x) \) of degree \( p - r \). If \( r \geq d \) then \( f(x) \) is apolar to \( g(u) \) whenever

\[
[h(u)g(u)|f(x)] = 0
\]

for all dual forms \( h(u) \) of degree \( r - p \).

Let \( \alpha \in \mathbb{C}^n \) and define the form

\[
(\alpha|x)^d := (\alpha_1 x_1 + \cdots + \alpha_n x_n)^d,
\]

\((\alpha|x)^d\) is a \( d \)-th power of a linear form. The method of apolarity, which will be presented in short, is useful to determine when a generic form of degree \( d \) can be written as a sum of an specific number of \( d \)-th powers of linear forms.

For \( c = (c_1, \ldots, c_n) \in \mathbb{C}^n \) define the operator \( D_{c,x} \) as

\[
D_{c,x} = c_1 \frac{\partial}{\partial x_1} + \cdots + c_n \frac{\partial}{\partial x_n}.
\]

For \( e = (e_1, \ldots, e_n) \in \mathbb{C}^n \), \( D_{c,x} \) and \( (e|x) \) are related by

\[
D_{c,x}(e|x)^d = \sum_{i=1}^n c_i \frac{\partial}{\partial x_i} (e_1 x_1 + \cdots + e_n x_n)^d
\]

\[
= \sum_{i=1}^d c_i e_i d (e_1 x_1 + \cdots + e_n x_n)^{d-1}
\]

\[
= d(e|c)(e|x)^{d-1}
\]

Using this equality we can establish the following results to express the notion of apolarity in terms of the action of \( D_{c,u} \) on a dual form.

**Proposition 1.** Let \( f(x) \) be a form of degree \( d - 1 \), and let \( g(u) \) be a dual form of degree \( d \). Then

\[
[g(u)|(c|f(x))] = [D_{c,u} g(u)|f(x)]
\]
Proposition 2. Let \( f(x) \) be a form of degree \( r \), and let \( g(u) \) be a dual form of degree \( d \), where \( r \leq d \). Then \( f(x) \) and \( g(u) \) are apolar iff

\[
[D_{c_{d-r}} \cdots D_{c_1} u] \cdot g(u) | f(x) | = 0
\]

for all \( c_1, \ldots, c_{d-r} \in \mathbb{C}^n \).

4. Main Theorem on Apolarity

Theorem 1. A generic \( f(x) \in H_d(\mathbb{C}^n) \) can be written in the form

\[
f(x) = (a|x)^d + (b|x)^d + \cdots + (c|x)^d
\]

iff there exists vectors \( a', b', \ldots, c' \in \mathbb{C}^n \) such that there is no non-zero dual form \( g(u) \in H^*_d(\mathbb{C}^n) \) apolar to all the forms \( (a'|x)^{d-1}, (b'|x)^{d-1}, \ldots, (c'|x)^{d-1} \in \mathbb{C}^n \).

As an illustration of the method provided by the main theorem we have the following corollary.

Corollary 1. A generic quaternary cubic can be written as the sum of 5 cubes.

Proof. Here \( n = 4 \) and \( d = 3 \). Choose the vectors \( a', b', \ldots, c' \) to be the vectors \( e_1, e_2, e_3, e_4, e_1 + e_2 + e_3 + e_4 \). Assume that a dual quaternary cubic \( g(u) \) exists which is apolar to all \( (e_1|x)^2, (e_2|x)^2, (e_3|x)^2, (e_4|x)^2, (e_1 + e_2 + e_3 + e_4|x)^2 \). Note that \( (e_i|x)^2 = x_i^2 \). If \( g(u) \) is apolar to \( x_i^2 \), then \( g(u) \) is also apolar to \( x_j x_i^2 \) for all \( j \) by definition of apolarity. Hence,

\[
0 = [g(u)|x_j x_i^2] = c_{u_j u_i^2} [u_j u_i^2|x_j x_i^2]
\]

where \( c_{u_j u_i^2} \) denotes the coefficient of \( u_j u_i^2 \) in \( g(u) \). Consequently \( c_{u_j u_i^2} = 0 \), so any term in \( g(u) \) containing a square must have coefficient equal to zero. Hence we can write \( g(u) \) in the form

\[
g(u) = c_1 u_2 u_3 u_4 + c_2 u_1 u_3 u_4 + c_3 u_1 u_2 u_4 + c_4 u_1 u_2 u_3
\]

But \( g(u) \) is also apolar to \( x_j (x_1 + x_2 + x_3 + x_4)^2 \). These conditions determine a system of linear equations with unknowns \( c_i \) whose coefficient matrix is

\[
\begin{pmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 1
\end{pmatrix}
\]

hence \( c_i = 0 \) for all \( i \). Thus \( g(u) = 0 \) and by the Main Theorem on apolarity we are done. \( \square \)
5. References

1. Richard Ehrenborg and Gian-Carlo Rota, *Apolarity and Canonical Forms for Homogeneous Polynomials*
4. Bruce Reznick, *On the length of binary forms*

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