BOOK EMBEDDINGS OF LAMAN GRAPHS

MICHELLE DELCOURT, MEGHAN GALIARDI, AND JORDAN TIRRELL

Abstract. We explore various techniques to embed planar Laman graphs in books. These graphs can be embedded as pointed pseudo-triangulations and have many interesting properties. We conjecture that all planar Laman graphs can be embedded in two pages.

1. Definitions

Definition 1. A graph $G$ with $n$ vertices and $m$ edges is a Laman graph if $m = 2n - 3$ and every subgraph induced by $k$ vertices (where $k > 1$) contains at most $2k - 3$ edges.

Definition 2. A book embedding of a graph $G$ is a drawing with no crossings of $G$ on a book; every edge is mapped to a page (a half-plane), and every vertex is mapped to the spine (the boundary shared by all the half-planes).

Definition 3. The book thickness of a graph $G$, denoted $bt(G)$ also known as page number, is the minimum number of pages needed to book embed $G$.

2. Finding Book Thickness Using Henneberg Constructions

Motivated by Schnyder labelings for triangulations, Huemer and Kappes proposed an analogous binary labeling for quadrangulations which decomposes the edge set in two directed trees rooted at a sink. A weak version of this labeling is valid for Laman graphs; however, the labeling does not guarantee a decomposition into two trees. The binary labeling of Laman graphs of Huemer and Kappes is based on Henneberg constructions. The Henneberg construction provides a decomposition into two trees whereas the labeling does not. A Henneberg construction consists of two moves:

I. Adding a vertex of degree two

II. Subdividing an edge, adding an edge adjacent to the new vertex [4]

Any Laman graph can be constructed from $K_3$ by a series of Henneburg moves. The order of moves is not always unique; some Laman graphs have multiple constructions.

This material is based upon work supported by grant DMS 0838434 “EMSW21MCTP: Research Experience for Graduate Students” and the National Science Foundation Graduate Research Fellowship under grant DGE 1144245.
A plane Laman graph $G$ can be decomposed into two trees by coloring the edges as $G$ is constructed using Henneberg constructions.\[3\] Start with the original triangle with vertices labeled, $v_0, v_1, v_2$. Vertices $v_0$ and $v_1$ will be the roots of the two trees, respectively $t_0$ and $t_1$. Two edges of $G$ will be colored black, while the other is colored gray. The black edges will be edges of $t_0$ while the gray edges will be edges of $t_1$. If the next move is a Henneberg I step, the new vertex will be connected to $t_0$ with a black edge and $t_1$ with a gray edge. Therefore this vertex will be a member of both trees. If the next move is a Henneberg II step, then the subdivided edge remains the same color and the new edge is colored depending on the vertex it is connected to. Thus, this vertex is only a member of one tree.

In our investigation we only got as far as looking at planar Henneberg I steps. Once $G$ is constructed using Henneberg constructions, the decomposition into two trees follows immediately. Conducting a depth first search on $t_0$ provides a permutation of vertices such that $t_0$ can be embedded on a single page. The problem arises when $t_1$ is embedded on the same permutation of vertices. To prove the conjecture that planar Laman graphs have book thickness 2, $t_1$ must have a single page embedding on the same permutation of vertices.

When constructing the Laman graph $G$, there can be many constructions for $G$, which decompose into different sets of trees. Once the $G$ is decomposed into the two trees, conducting a depth first search on one of the trees can also yield a variety of different permutations for the vertices depending on which branch of the tree is traversed first. For all planar Laman graphs using Henneberg I steps we have looked at, we have found that there exists a construction such that there also exists a permutation of vertices corresponding to a depth first search of one of the trees such that the graph can be embedded in 2 pages. However, this is not a proof that planar Laman graphs constructed using Henneberg I steps have book thickness 2, but we have yet to find a counter example of a planar Laman graphs constructed using Henneberg I steps that have book thickness 3.

3. Finding Book Thickness Using Separating Triangles

Another approach is to use the separating triangles in the Laman graph to create a book embedding. Every plane graph with no separating triangle is a subgraph of a Hamiltonian plane graph.\[5\] Any graph which is a subgraph of a Hamiltonian plane graph has book thickness 2.\[2\]

Let $G$ be a planar Laman graph. If $G$ has no separating triangle then $G$ has book thickness 2. Kaufmann and Wiese present a technique is presented that takes planar graphs with separating triangles and adds dummy vertices to remove the separating triangles.\[6\] This technique also causes the graph
to become 4-connected. This technique can be applied to a Laman graph \( G \) that contains separating triangles and results in a graph \( G' \). If a planar graph is 4-connected then a Hamiltonian cycle can be found (in linear time too).[1]

Since \( G' \) is now 4-connected, \( G' \) is Hamiltonian by a Theorem by Tutte. Therefore \( G' \) has book thickness 2. The order vertices of \( G' \) appear in the Hamiltonian is the order on the spine that leads to a book embedding with thickness 2. Since the Hamiltonian cycle must go through all vertices of \( G' \), the dummy vertices also appear in the book embedding. In order to have a book embedding for \( G \), the dummy vertices must be removed. For the removal of each dummy vertex, four edges must be deleted from the book embedding and one edge must be added. The removal of edges from a book embedding cannot increase the book thickness. Adding an additional edge to the book embedding can increase the book thickness by at most one. Therefore the book thickness of \( G \) increases at most linearly with each separating triangle. Since it is known that all planar graphs have book thickness of four or less, [7] our result is only useful for a Laman graph with one separating triangle. We are able to say that a planar Laman graph with one separating triangle has book thickness of at most 3 (although we still believe the book thickness of plane Laman graphs to be 2).

4. Further research

**Conjecture 1.** If \( G \) is a planar Laman graph, then \( bt(G) \leq 2 \).

Is there a stronger argument for planar Laman graphs with separating triangles? Another open problem is whether the book thickness of non planar Laman graphs can be bounded.

5. Bibliography

**References**


Michelle Delcourt, Department of Mathematics, University of Illinois at Urbana-Champaign
E-mail address: delcour2@illinois.edu

Meghan Galiardi, Department of Mathematics, University of Illinois at Urbana-Champaign
E-mail address: galiard2@illinois.edu

Jordan Tirrell, Department of Mathematics, Brandeis University
E-mail address: jтирrell@brandeis.edu