The hyperkähler geometry of the deformation space of complex projective structures on a surface

Brice Loustau

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The hyperkähler geometry of $\mathbb{CP}(S)$

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Outline

1. Complex projective structures
2. The character variety
3. The Schwarzian parametrization
4. The minimal surface parametrization
1 Complex projective structures

2 The character variety

3 The Schwarzian parametrization

4 The minimal surface parametrization
What is a complex projective structure?
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Let $S$ be a closed oriented surface of genus $g \geq 2$. 
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Definition

A complex projective structure on $S$ is a $(G, X)$-structure on $S$ where the model space is $X = \mathbb{CP}^1$ and the Lie group of transformations of $X$ is $G = \text{PSL}_2(\mathbb{C})$. 
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\[ \gamma = \frac{az + b}{cz + d} \in G = \text{PSL}_2(\mathbb{C}) \]
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$$\mathcal{CP}(S) = \{\text{all } \mathbb{C}P^1\text{-structures on } S\}/\text{Diff}^+_0(S).$$

A point $Z \in \mathcal{CP}(S)$ is called a marked complex projective surface.
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The character variety
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There is a forgetful map $p : \mathcal{CP}(S) \to \mathcal{T}(S)$ where

$$\mathcal{T}(S) = \{\text{all complex structures on } S\}/\text{Diff}^+_0(S)$$

is the Teichmüller space of $S$. 
Fuchsian and quasifuchsian structures
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If any Kleinian group $\Gamma$ (i.e. discrete subgroup of $PSL_2(\mathbb{C})$) acts freely and properly on some open subset $U$ of $\mathbb{CP}^1$, the quotient inherits a complex projective structure.
Fuchsian and quasifuchsian structures

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\mathbb{C}P^1 & \xrightarrow{\Gamma \subset PSL_2(\mathbb{C})} S
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Fuchsian structures
In particular, any Riemann surface $X$ can be equipped with a compatible $\mathbb{CP}^1$-structure by the uniformization theorem:

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**Note:** This defines a *Fuchsian section* $\sigma_{\mathcal{F}} : \mathcal{T}(S) \to C\mathbb{P}(S)$. 
Quasifuchsian structures
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Complex projective structures

The character variety

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Holonomy
Any complex projective structure \( Z \in \mathcal{CP}(S) \) defines a *holonomy representation* \( \rho : \pi_1(S) \to G = \text{PSL}_2(\mathbb{C}) \).
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Holonomy defines a map

$$\text{hol} : \mathcal{CP}(S) \rightarrow \mathcal{X}(S, G) ;$$

where $\mathcal{X}(S, G) = \text{Hom}(\pi_1(S), G) // G$ is the character variety of $S$. 
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By a general construction of Goldman, the character variety $\mathcal{X}(S, G)$ enjoys a natural complex symplectic structure $\omega_G$. 
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Abusing notations, we also let $\omega_G$ denote the complex symplectic structure on $CP(S)$ obtained by pulling back $\omega_G$ by the holonomy map $hol : CP(S) \rightarrow \mathcal{X}(S, G)$.
The character variety (continued)
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<th>Theorem (Goldman)</th>
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$$\sigma_{\mathcal{F}}^*(\omega_G) = \omega_{WP}.$$  

**Theorem (Platis, L)**

Complex Fenchel-Nielsen coordinates $(l_i, \tau_i)$ associated to any pants decomposition are canonical coordinates for the symplectic structure:

$$\omega_G = \sum_{i} dl_i \wedge d\tau_i.$$
Hitchin-Kobayashi correspondence
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Theorem (Hitchin, Simpson, Corlette, Donaldson)

Fix a complex structure $X$ on $S$. There is a real-analytic bijection

$$H_X : \mathcal{X}^0(S, G) \xrightarrow{\sim} \mathcal{M}_{\text{Dol}}^0(X, G)$$

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**Theorem (Hitchin)**

There is a natural hyperkähler structure $(g, I, J, K)$ on $\mathcal{M}^0_{\text{Dol}}(X, G)$. The map $H_X$ is holomorphic with respect to $J$. It is also a symplectomorphism for the appropriate symplectic structures.
1. Complex projective structures
2. The character variety
3. The Schwarzian parametrization
4. The minimal surface parametrization
The cotangent hyperkähler structure
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**Theorem (Feix, Kaledin)**

If $M$ is a real-analytic Kähler manifold, then there exists a unique hyperkähler structure in a neighborhood of the zero section in $T^*M$ such that:

- it refines the complex symplectic structure
- it extends the Kähler structure off the zero section
- the $U(1)$-action in the fibers is isometric.
The Schwarzian parametrization
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- It turns a fiber $\sim p^{-1}(X)$ into a complex affine space modeled on the vector space $H^0(X, K^2) = T^*_X \mathcal{T}(S)$. 


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For each choice of $\sigma$, we thus get a symplectic structure $\omega^\sigma$ on the whole space $\mathbb{CP}(S)$ (pulling back $\omega_{\text{can}}$) and a hyperkähler structure on some neighborhood of the Fuchsian slice.
The Schwarzian parametrization (continued)
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**Theorem (L)**

\[ \mathcal{CP}(S) \approx^\sigma T^* T(S) \text{ is a complex symplectomorphism iff } \]
\[ d(\sigma - \sigma_F) = \omega_{WP} \text{ (on } T(S)) \].
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Using results of McMullen (also Takhtajan-Teo, Krasnov-Schlenker):

### Theorem (Kawai, L)

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  - This is similar to the situation we saw with the Hitchin-Kobayashi correspondence. Quiz: what is a significant difference though?
1 Complex projective structures

2 The character variety

3 The Schwarzian parametrization

4 The minimal surface parametrization
The minimal surface parametrization
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The space of almost-Fuchsian structures $\mathcal{AF}(S) \subset Q\mathcal{F}(S)$ is a neighborhood of the Fuchsian slice such that if $Z \in \mathcal{AF}(S)$, the hyperbolic 3-manifold associated to $Z$ contains a unique minimal surface $\Sigma$. 
The minimal surface parametrization

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The Gauss-Codazzi equations satisfied by the second fundamental form $II_\Sigma$ are equivalent to the fact that $II_\Sigma$ is the real part of a unique holomorphic quadratic $\varphi$. 
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This defines a map

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It is a diffeomorphism of $\mathcal{AF}(S)$ onto some neighborhood of the zero section of $T^*T(S)$. 
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Again, one can use this “minimal surface parametrization” to pull back the hyperkähler structure of $T^*\mathcal{T}(S)$ on $\mathcal{CP}(S)$. 
The minimal surface parametrization (continued)
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Using arguments of Krasnov-Schlenker to compute the variation of $W$ under an infinitesimal deformation of the metric, one shows:

**Theorem (L)**

The minimal surface parametrization $\mathcal{AF}(S) \xrightarrow{\sim} T^*T(S)$ is a real symplectomorphism (for the appropriate symplectic structures).