Some Solutions, Problem Set #1

Prove that
\[ f(s) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \prod_p \left( 1 - \tau(p) p^{-s} + p^{1-2s} \right)^{-1}, \quad s > \frac{13}{2}. \]

Let
\[ f_p(s) = \sum_{n=1}^{\infty} \frac{\tau(p^n)}{p^{ns}}. \]

Since \( \tau(n) \) is multiplicative,
\[ f(s) = \prod_p \left( 1 + f_p(s) \right). \]

Thus, we must show that
\[ 1 + f_p(s) = \left( 1 - \tau(p) p^{-s} + p^{1-2s} \right)^{-1}. \]

We use Ramanujan's recurrence relation
\[ \tau(p^{n+1}) = \tau(p) \tau(p^n) - p^{1-2s} \tau(p^n), \quad n \geq 1, \]

to deduce that
\[ f_p(s) = \frac{\tau(p)}{p^s} + \sum_{n=2}^{\infty} \frac{\tau(p^n)}{p^{ns}} = \frac{\tau(p)}{p^s} + \sum_{n=2}^{\infty} \left( \frac{\tau(p) \tau(p^{n-1})}{p^{ns}} - \frac{p^{1-2s} \tau(p^{n-2})}{p^{ns}} \right) \]
\[ = \frac{\tau(p)}{p^s} + \frac{\tau(p)}{p^s} \sum_{n=1}^{\infty} \frac{\tau(p^n)}{p^{ns}} - \frac{p^{1-2s}}{p^{2s}} \sum_{n=0}^{\infty} \frac{\tau(p^n)}{p^{ns}} \]
\[ = (\tau(p)p^{-s} + \tau(p)p^{-s}) f_p(s) - p^{1-2s} f_p(s) - p^{1-2s}, \]

Solving for \( f_p(s) \), we deduce that
\[ f_p(s) = \frac{\tau(p)p^{-s} - p^{1-2s}}{1 - \tau(p)p^{-s} + p^{1-2s}}, \]
or
\[ 1 + f_p(s) = \frac{1 - \tau(p)p^{-s} + p^{1-2s} + \tau(p)p^{-s} - p^{1-2s}}{1 - \tau(p)p^{-s} + p^{1-2s}} \]
\[ = \frac{1}{1 - \tau(p)p^{-s} + p^{1-2s}}, \]

and so (x) has been proved.
Prove that \( \tau_2(n) > 0 \) iff each prime congruent to 3 modulo 4 in the canonical factorization of \( n \) appears with an even exponent.

Recall

\[
\tau_2(n) = 4 \sum_{d \text{ odd}} (-1)^{(n-1)/2} d^{(n-1)/2} \tag{X}
\]

It is easily checked that

\[
f(n) = \begin{cases} 
(-1)^{(n-1)/2}, & n \text{ odd,} \\
0, & n \text{ even,}
\end{cases}
\]

that \( f(n) \) is multiplicative (in fact, completely multiplicative). Then, from elementary number theory, \( \tau_2(n) \) is multiplicative. If

\[
N = \prod_{i=1}^{r} p_i^a_i \prod_{j=1}^{s} q_j^b_j, \quad p_i \equiv 1 \pmod{4}, \quad q_j \equiv 3 \pmod{4}
\]

then, by multiplicativity,

\[
\tau_2(N) = \prod_{i=1}^{r} \tau_2(p_i^a_i) \prod_{j=1}^{s} \tau_2(q_j^b_j).
\]

Clearly \( \tau_2(p_i^a_i) > 0 \) from \( (X) \) as all divisors of \( p_i^a_i \) are congruent to 1 modulo 4. If \( b_j \) is even, we note that there is one more divisor \( \equiv 1 \pmod{4} \) than \( \equiv 3 \pmod{4} \), i.e., \( \tau_2(q_j^{b_j}) > 0 \). However, if \( b_j \) is odd, then \( d_{1,4}(q_j^{b_j}) = d_{3,4}(q_j^{b_j}) \), i.e., \( \tau_2(q_j^{b_j}) = 0 \). Thus, \( \tau_2(N) > 0 \) iff \( b_j \) is even, \( 1 \leq j \leq s \).
For $|q| < 1, |y| < 1$, show (without using Ramanujan's $\psi$-summation formula) that

$$\sum_{n=-\infty}^{\infty} \frac{x^n}{1-yq^n} = \sum_{n=-\infty}^{\infty} \frac{y^n}{1-xq^n}.$$ 

We have

$$\sum_{n=-\infty}^{\infty} \frac{x^n}{1-yq^n} = \sum_{n=0}^{\infty} \frac{x^n}{1-yq^n} + \sum_{n=1}^{\infty} \frac{x^{-n}}{1-yq^{-n}} = \sum_{n=0}^{\infty} \frac{x^n}{1-yq^n} - \sum_{n=1}^{\infty} \frac{y^{-1}(q/x)^n}{1-yq^n}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} y^m \frac{x^{n+m}}{1-yq^n} - \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} \frac{y^{-m}}{y^{m+1}} \frac{x^{n-m-1}(m+1)n}{1-yq^n}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} y^m \frac{x^{n+m}}{1-yq^n} - \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{y^{-m}}{y^{m}} \frac{x^{n-m}}{1-yq^n},$$

which is symmetric in $x$ and $y$, since the order of summation can be inverted by absolute convergence.

Hence,

$$\sum_{n=-\infty}^{\infty} \frac{x^n}{1-yq^n} = \sum_{n=-\infty}^{\infty} \frac{y^n}{1-xq^n}.$$